Comments on Kevin Richardson's "Logical Subtraction as Relevant Implication"

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Thanks, Kevin — what a cool and interesting paper! It raises some really important questions:

> *What is the logic of subtraction?*
> *What is the relationship between subtraction and conditionals?*
> *Is there more than one legitimate conception of logical subtraction? (And if so: which ones?)*
> *What should be the aim of a theory of logical subtraction? What should it settle?*
> *Should the remainder $A - B$ always be defined for any propositions A and B?*

Only the first two questions have received much detailed discussion in the literature, but I think they all really repay explicit discussion. So one great contribution of your paper is just to put all this on the table, and draw the connections. In my comments I'll mostly speak to the first question and the last two.

One important line of argument underpinning your approach is something like the following:

P1) A theory of subtraction should give us a well-behaved *logic of subtraction*.
P2) SY and DH's theories of subtraction don't give us such a logic.
P3) KR's theory of subtraction does yield such a logic.
C) We should adopt KR's theory over SY and DH's.

Now I won't take issue with (P2) — I think you are quite right to point out this limitation of our accounts. But I do want to interrogate (P1) and (P3) a little more. My questions about (P3) are a bit nit-picky: I'm not convinced that your theory, as articulated in the paper, works as advertised. My question about (P1) is more big picture. One of your core complaints about SY and DH is that we do not guarantee the existence of a well-defined remainder $A - B$ for any given A and B. But we actually regard this as a virtue of our theory, not a bug! We take it to be one of the tasks of a theory of subtraction to separate cases where B is a separable part of A from cases where B is inseparable from A. At least the face of it, your account leaves that work undone, because it makes subtraction a total operator.

Zooming in, what do we mean by "a well-behaved logic of subtraction"? I take it that having such a logic is a matter of settling questions about the logical behaviour of the subtraction operator. For one, it should settle whether or not equivalences like the following hold true in general (and if not in general, under what conditions they do hold):

1. $(A \wedge B) - B \quad = \quad A$
2. $(A - B) \wedge B \quad = \quad A$
3. $(A - B) \wedge B \quad = \quad (A \wedge B)$
4. $A - A \quad = \quad \top$

5. $(A - B) - C \quad = \quad (A - C) - B$
6. $A - (B \wedge C) \quad = \quad (A - B) - C$
7. $A - (A \vee B) \quad = \quad \bot$
8. $\bot - A \quad = \quad \neg A$

More substantively, the word "well-behaved" suggests a requirement to the effect that some particular subset of these logically principles *do* (at least typically) hold according to the theory. In particular, it seems reasonable to require that some suitably restricted versions of (1-4) should hold in order for an operator to deserve the name "subtraction": after all, the basic conception of subtraction is as a kind of inverse to conjunction.

The heart of your criticism against Yablo and me is that we do not convincingly meet these requirements. And I basically think you're right about that. As you have explained, Yablo's non-recursive conception of truth maker semantics is so underspecified that it doesn't really clear the first bar. That is to say: we don't have enough of a hold on the theory to decide with any precision whether/when any of these equivalences hold true.

On my own theory, as you point out, these logical questions don't even make straightforward sense, as subtraction isn't a binary operator for me (esp. questions about iterative subtraction, like (5) and (6)). Moreover, to the extent that you can transpose these claims, the subject matter-dependence in my theory means that for many such equations, the question of whether they hold has an unsatisfying answer: viz., it depends on context. In my defence, I do want to point out that (3) and (4) are exceptions to this rule: these hold true for me in any context where the remainder is defined. Consequently (2) is also validated when it can be: viz. whenever A entails B. That provides *some* justification for saying that I really am talking about a kind of subtraction. But I grant that someone could quite reasonably want more than this, and especially a more robust vindication of (1).

What about your theory? It's certainly going to clear the first bar: we should be able to tell from your semantic entries whether or not the LHS and the RHS of these equations have the same verifiers. But going from the semantic entries you have on p. 7, it does not seem to me like it validates the most important ones… Take your example form the paper: *Sparky is a brown dog – Sparky is brown*. You would expect the remainder to be *Sparky is a dog*, in line with (1). However, this is not what we get. In your model, *Sparky is a dog* has only one verifier, **dog**. But note that the fusion **dog** $\sqcup$ **brown** is a verifier of the remainder *Sparky is a brown dog – Sparky is brown* as well: for this state can be fused with any verifier of *Sparky is brown* to get a verifier of *Sparky is a brown dog*. Consequently, *Sparky is a dog* does not have the same verifiers as *Sparky is brown dog – Sparky is brown*, on your account.

Generalising, (1) is strongly invalidated on your theory: for almost any A and B, it will be the case that $(A \wedge B) - A \neq A$. Moreover, (2-4) are also invalidated as far as I can see. Now since you don't secure any of (1-4), that seems to me to threaten your claim to having a well-behaved logic of subtraction…[1] It's worth noting though that your account *does* mostly validate (5-7).

Still, on balance, your *Remainder* clause seems to me like a bit of a non-starter. However, that is by no means an indictment of the general approach. It's a technical problem, which may have a technical fix. One thing that could help you is to add a minimality condition to the clause: just say **s** verifies $A - B$ iff (i) **s** can be fused with any verifier of B to make a verifier of A and (ii) ***no proper part of* s *also has that property***. Moreover, Jago's analysis of relevant implication may have some resources to address these technicalities.

Whatever the details, let us suppose these issues are fixed. Then we come to a more interesting question. A signature feature of your approach is that you take the remainder $A - B$ to be well-defined for any propositions A and B. This really sets your account apart from other theories of subtraction, which have typically taken the partiality of subtraction as a given.

I think this property of your account is exciting, suggesting a new perspective on puzzles about subtraction. But it does make me wonder what happens, on the picture you favour, to the intuitive distinction between possible and impossible subtractions. In English, the effect of subtraction $P - Q$ can be captured by saying "*P* except maybe not *Q*." We can sometimes use this sort phrase to retract part of our earlier assertion without any contradiction. For example:

9.	*The shop is open every day,*	*except that*	*it might not be open on Sundays.*
10.	*Pescatarians are vegetarians,*	*except that*	*they might eat fish.*
11.	*A gratin is a quiche,*	*except that*	*it might not be baked in a shell.*

In these cases, the remainder seems intuitively clear and well-defined. But in other cases, we try to do the same thing and the result is nonsense:

12.	*Joan is in Thailand,*	*except that*	*she is not in Asia.*
13.	*Holly swam very slowly,*	*except that*	*she did not swim at all.*
14.	*Fred is thirsty,*	*except that*	*nobody is thirsty.*
15.	*"The Hobbit" is a true story,*	*except that*	*no hobbit has ever existed.*

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[1] (2) is invalid just because you allow the remainder $A - B$ to be defined when A does not entail B. For (3), a counterexample would be $((P \vee Q) - P) \wedge P \neq (P \vee Q) \wedge P$. For suppose the only contingent states are **p**, **q** and $\mathbf{p}\sqcup\mathbf{q}$, with **p** and **q** disjoint. Also suppose *P*'s only verifier is **p** and Q's only verifier is **q**. Then given your semantic entries on p. 7, we get:

> $(P \vee Q)$ has the verifiers **p** and **q**
> $(P \vee Q) \wedge P$ has the verifiers **p** and $\mathbf{p}\sqcup\mathbf{q}$
>
> $(P \vee Q) - P$ has only one verifier, **p**
> $((P \vee Q) - P) \wedge P$ has only one verifier, **p**

Hence $((P \vee Q) - P) \wedge P \neq (P \vee Q) \wedge P$, which makes for a counterexample to (3). The underlying source of weirdness here is that on your semantics $((P \vee Q) - P) = P$, which is counterintuitive. As for (4): all verifiers of A are verifiers of $A - A$. But only the empty state is a verifier for $\top$. (Or, on an alternative conception of top, every state is a verifier of $\top$. But then it's still true that $A - A \neq \top$.)

In these cases, subtraction is intuitively not possible: the remainder is apparently undefined. That's a basic reason, I think, why people have typically assumed that the subtraction operator had to be partial. That is to say, a motivation for having a partially defined subtraction operator is to capture and explain the difference between (9-11) on the one hand, and (12-15) on the other hand. Yet on the face of it you seem to deny the existence of this distinction, since on your account any proposition can be subtracted from any other…

This raises a number of questions. On the picture you favour, is there a different way of capturing the contrast between (9-11) and (12-15)? If you do a "weird" subtraction like (14), what proposition do you get out? That is to say, what are the verifiers for the remainder of *Fred is thirsty — Somebody is thirsty*? One thought I had is that maybe the equivalence (7) points the way. On your account it's *possible* to subtract a disjunction from its disjunct, but the result is (typically) a contradiction. Maybe it's the same with (12-15): these sentences strike us as nonsensical because they're *contradictory*, not because they're *undefined*. That potentially makes for very fresh perspective on the puzzle that (9-15) pose.

Besides wanting to explain the difference between (9-11) and (12-14), another motivation for delineating the range of permissible remainders is to adjudicate disputes about philosophically controversial subtractions:

16.	*Justified belief is knowledge,*	*except that*	*it may not be true.*
17.	*Einstein's theory of relativity is true,*	*except that*	*mathematical objects may not exist.*
18.	*The King of France is bald*	*except that*	*France has no King.*
19.	*Lying is asserting an intentional falsehood*	*except that*	*it may be true by accident.*
20.	*Willing your arm to go up is raising your arm,*	*except that*	*your arm may not go up.*

In these cases, it's controversial whether the subtraction in question is possible. For instance, Mark Colyvan criticises "easy-road" nominalism precisely on the grounds that (17) is just as illegitimate as (15). How to adjudicate such claims if all subtractions are well-defined?

You do make a tantalising suggestion about this, when you say that we can subtract the maths from mixed mathematical statements by subtracting the **abstract** part of their verifiers, which are of the form **concrete** ⊔ **abstract**. This suggests that for you there are, after all, conditions that are especially conducive to subtraction, and consequently a distinction between more and less successful subtractions. Could you say more about that distinction? Also, I'd love to hear your thoughts about how you could tell, in practice, whether conditions are good for subtraction. For instance, how could you check whether the states that verify Einstein's theory are in fact nicely separable into **concrete** and **abstract** components?