# Comments on Kevin Richardson's "Logical Subtraction as Relevant Implication"

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*A full write-up of these comments is available at danielhoek.com.*

## The Account

A Fine/Jago style truth-maker semantics with standard clauses for conjunction and disjunction, combined with the following clause for remainder:

> ***Remainder***. A state **s** verifies $A - B$ iff for every verifier **b** of $B$, **s** $\sqcup$ **b** verifies $A$.

Some of the questions raised: *What is the logic of subtraction? What should be the aim of a theory of logical subtraction? Should the remainder A – B always be defined for any propositions A and B?*

## Argument

One line of argument in the paper runs like this:

P1) A theory of subtraction should give us a well-behaved *logic of subtraction*.
P2) SY and DH's theories of subtraction don't give us such a logic.
P3) KR's theory of subtraction does yield a well-behaved logic of subtraction.
C) We should adopt KR's theory over SY and DH's.

I don't really take any issue with (P2) — Kevin's criticism here points to real limitations of these accounts! But I do have some more questions about (P1) and (P3)…

## Logical Principles of Subtraction

Some potential logical principles of subtraction include the following:

1. $(A \wedge B) - B = A$
2. $(A - B) \wedge B = A$
3. $(A - B) \wedge B = (A \wedge B)$
4. $A - A = \top$
5. $(A - B) - C = (A - C) - B$
6. $A - (B \wedge C) = (A - B) - C$
7. $A - (A \vee B) = \bot$
8. $\bot - A = \neg A$

*Sidenotes:*

A. Equivalences (1) and (2) capture the idea of subtraction as the inverse of conjunction. If subtraction is a total operator, (2) can't hold in general, since it implies $A$ entails $B$. (3) is the natural generalisation of (2), implying (2) in the special case where $B$ is a conjunctive part of $A$ (i.e. the case where $A \wedge B = A$).

B. (5) and (6) are intuitive iterative principles of subtraction, with (6) implying (5).

C. (7) could be endorsed as capturing the idea that you cannot subtract a disjunction from its disjuncts. While (7) is inconsistent with (1), (2) and (3), it is validated by Kevin's *Remainder* clause: unless $B$ happens to be part of $A$, no state will combine with every verifier of $B$ to make a verifier for $A$.

D. In the truthmaker framework, multiple versions of "$\top$" and "$\bot$" are sometimes distinguished, yielding different versions of (4), (7) and (8). On the assumption that $(\top \wedge A) = A$, (1) implies (4).

## How did we do?

- **SY** claims to endorse versions of (1) and (2), but his account of non-recursive semantics is too impressionistic to determine the bounds of their validity with any great precision. For the same reason, it's not clear where the other principles fall on his account.
- **DH** does not treat subtraction as a binary operator. Keeping subject matter constant, the account validates (3) and hence (2) in the special case where *A* entails *B*; (1) only holds if *A* is relevant and the remainder defined. Also validates (4); invalidates (5-8).
- **KR** invalidates (1-4), but validates (5-7).

## Impossible Subtractions and Partiality

In natural language, (*P* – *Q*) may be rendered as "*P*, except maybe not *Q*".

Some **possible subtractions** are:

| | | | |
|---|---|---|---|
| 9. | *The shop is open every day,* | *except that* | *it might not be open on Sundays.* |
| 10. | *Pescatarians are vegetarians,* | *except that* | *they might eat fish.* |
| 11. | *A gratin is a quiche,* | *except that* | *it might not be baked in a shell.* |

Some **impossible subtractions** include:

| | | | |
|---|---|---|---|
| 12. | *Joan is in Thailand,* | *except that* | *she is not in Asia.* |
| 13. | *Holly swam three laps,* | *except that* | *she did not swim at all.* |
| 14. | *Fred is thirsty,* | *except that* | *nobody is thirsty.* |
| 15. | *"The Hobbit" is a true story,* | *except that* | *no hobbit has ever existed.* |

*Question:* How to capture the distinction between (9-11) and (12-15) if subtraction is a complete operator?

*Possible answer:* (12-15) are *contradictory* rather than undefined.

**Controversial/philosophical cases**:

| | | | |
|---|---|---|---|
| 16. | *Justified belief is knowledge,* | *except that* | *it may not be true.* |
| 17. | *Einstein's theory of relativity is true,* | *except that* | *mathematical objects may not exist.* |
| 18. | *The King of France is bald* | *except that* | *France has no King.* |
| 19. | *Lying is asserting an intentional falsehood* | *except that* | *it may be true by accident.* |
| 20. | *Willing your arm to go up is raising your arm,* | *except that* | *your arm may not go up.* |