# Flipping Coins, Spinning Tops and the Continuum Hypothesis 

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#### Abstract

"The totality of our so-called knowledge or beliefs, from the most casual matters of geography and history to the profoundest laws of atomic physics or even of pure mathematics and logic, is a manmade fabric which impinges on experience only along the edges ... Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system. ...Conversely, by the same token, no statement is immune to revision. Revision even of the logical law of the excluded middle has been proposed as a means of simplifying quantum mechanics; and what difference is there in principle between such a shift and the shift whereby Kepler superseded Ptolemy, or Einstein Newton, or Darwin Aristotle?" (Quine, Two Dogmas, 39-40)


Quine's Thesis. Even our views about logic and mathematics ought to be sensitive to empirical evidence.

My Claim. The presuppositions underlying the probabilistic inductive methods standardly used in science entail that Cantor's continuum hypothesis is false. Thus the success of those methods provides us with a powerful, empirical reason to reject the continuum hypothesis.

## Infinite Cardinalities

The Pigeon-Hole Principle. Two collections of objects have the same cardinality just in case there is a one-to-one correspondence relating the members of the first collection to the members of the second.

Countable infinity ( $\aleph_{0}$ ) is the number of...

- Miles you will walk if you start walking at a constant speed and never stop
- Hours you will walk if you walk that walk
- Minutes you will walk if you walk that walk
- Miles you will walk if you start walking at double that speed and never stop
- Natural numbers (0, 1, 2, 3 ...)
- Integers (... $-3,-2,-1,0,1,2,3 \ldots$ )
- Rational numbers $\left(\ldots-1 / 3 \ldots 0 \ldots 1 / 2 \ldots 2 / 3 \ldots 1 \ldots 1^{1 / 5} \ldots\right)$

The Continuum ( $2^{\aleph_{0}}$ ) is the number of...

- Different ways $\aleph_{0}$ coins might land (HTHTHTTTH..., TTHTHHHTT..., ...)
- Angles at which a spinning top might land
- Points on a continuous line or interval
- Infinite decimal representations/real numbers (0.33333...; 1.61803... ; 2.71828...)
- Functions from natural numbers to natural numbers

Aleph-1 $\left(\aleph_{1}\right)$ is the first cardinality greater than $\aleph_{0}$, and the number of countable ordinals. (And more generally, $\aleph_{\alpha+1}$ is always the first cardinal greater than $\aleph_{\alpha}$, a.k.a. its successor.)

Cantor's Theorem: $2^{\aleph_{0}}>\aleph_{0}$
Continuum Hypothesis (CH): $2^{\aleph_{0}}=\aleph_{1}$

ZFC (Zermelo-Fraenkel Set Theory with Choice) is an axiomatic first-order theory. Most of modern mathematics, including all the analysis and algebra that physical theories use, can in principle be carried out "within" ZFC.

Independence of CH from ZFC: Assuming ZFC is consistent, so are ZFC + CH (Gödel 1938) and ZFC $+\neg \mathrm{CH}$ (Cohen 1963). So the continuum hypothesis cannot be proved or disproved in ZFC. If you add large cardinal axioms to ZFC, CH remains undecided (Levy and Solovay 1967, Honzik 2017).

Cardinal properties you needn't worry about at all. A cardinal $\mathcal{K}$ is:

- A limit cardinal iff $\kappa$ is not the immediate successor of any cardinal (such as $\aleph_{0}$ ).
- Regular iff it is the limit of less than $\mathcal{\kappa}$ cardinals less than $\kappa$. It is singular iff it is not regular ( $\aleph_{\omega}$ is an instance of a singular cardinal).
- Real-valued measurable* iff a total, $\kappa$-additive probability measure can be defined on $\mathcal{\kappa}$.
- Measurable* (or two-valued measurable) iff a total measure can be defined on $\kappa$ that has the doubleton $\{0,1\}$ as its range and is $\lambda$-additive for any $\lambda<\kappa$.
- Weakly inaccessible* iff it is a regular limit cardinal.
- Weakly Mahlo* iff every closed unbounded subset of $\kappa$ contains a weakly inaccessible cardinal.
- Strongly inaccessible iff it is regular and for all $\lambda<\kappa, 2^{\lambda}<\kappa$.

The starred properties are large cardinal properties, which is to say that ZFC does not prove there are any cardinals that instantiate them.

## The Main Argument

Premises. Random process $\mathbf{R}$ has a chance distribution $\mathrm{Ch}: P(\Omega) \rightarrow[0,1]$ with the following properties:
A. Continuum of Outcomes: $\# \Omega=2^{\aleph_{0}} ; \operatorname{Ch}(\Omega)=1$.
B. Zero-Chance Outcomes: $\operatorname{Ch}(\{x\})=0$ for every $x \in \Omega$.
C. Totality: There is a chance $\operatorname{Ch}(E)$ associated with every event $E \subseteq \Omega$.
D. Countable Additivity: If $E_{0}, E_{1}, \ldots \subseteq \Omega$ are all disjoint, then $\operatorname{Ch}\left(E_{0} \cup E_{1} \cup \ldots\right)=\operatorname{Ch}\left(E_{0}\right)+\operatorname{Ch}\left(E_{1}\right)+\ldots$

Conclusions. Given the axioms of ZFC, if a function with the properties (A-D) exists, it follows that
I. $2^{\aleph_{0}}>\aleph_{1}$ (Banach and Kuratowski 1929)
II. $2^{\kappa_{0}}$ exceeds a weakly inaccessible cardinal (Ulam 1930)
III. $2^{\aleph_{0}}$ exceeds a weakly Mahlo cardinal (Solovay 1997)

The argument is schematic. Candidate random processes $\mathbf{R}$ include:

- flipping countably infinitely many coins; flipping a single coin infinitely many times
- spinning a top/spinner/roulette wheel; throwing a random dart at a dartboard
- conducting countably many spin measurements on a particle in superposition
- performing a momentum measurement on a particle with known position.

Formally, I propose to adding to ZFC the following axiom:
M. The continuum admits a countably additive, total measure without any positivevalued singletons.
ZFC +M is equiconsistent with ZFC + "There is a measurable cardinal" (Solovay 1997).

## A. Continuum of Outcomes

Objection 1. A realistic spinner does not pick out any one determinate point.
Reply. It does as long as we adopt a clever convention for what counts as the 'point picked out'.
And anyway, it is hard to see how this objection touches the coin flip cases.
Objection 2. The spinner only picks out a point on the continuum if space is actually continuous.
Reply. Actually, if space turned out to be dense but not continuous the procedure just specified is not guaranteed to pick out a spatial point at all: it would pick out a Dedekind cut, and there are still continuum many of those.
And anyway, the number of outcomes of a coin flip is not contingent on the structure of space.

## B. Zero-Chance Outcomes

Let $x$ be any outcome of $\mathbf{R}$, then
B1. For any positive natural number $n$, there is an event $E$ containing $x$ such that $\mathrm{Ch}(E) \leq 1 / 2^{n}$.
B2. $\quad$ The chance of $x$ is a non-negative real number.
$\therefore \mathrm{B}^{*}$. $\mathrm{Ch}(\{x\}) \leq 1 / 2^{n}$ for every $n \quad$ (from B1 and finite additivity of Ch )
$\therefore$ B. $\operatorname{Ch}(\{x\})=0 \quad$ (from B2 and B*)

Objection 3. What if the chances were infinitesimally small but non-zero? (Benci et al. 2013)
Reply. It doesn't matter. Even if chances are hyperreals, we can round those values off to the nearest real number, and that gives us the real-valued function we need for the argument.

Objection 4. Determinism is true, so in fact all events and outcomes either have chance 1 or chance 0. Reply. Even a determinist ought to be hesitant to be an antirealist about non-trivial chances. And] there are perfectly deterministic articulations of the notion of chance out there (e.g. Eagle 2011, Gallow ms). And anyway, our best theory of nature, Quantum Mechanics, posits chances at the fundamental physical level. (For QM analogues of the coin flip situation or the spinner see above).

## C. Totality

C1. Scientists have found the following form of inference to be fruitful and reliable: "During our $n$ independent runs of the random procedure we observed $k$ occurrences of the event $E$. Therefore $\operatorname{Ch}(E) \approx k / n, "$ imposing no preconditions on the event $E$.
C2. This inference presupposes that $E$ has a chance.
C3. The best explanation for C 1 is that the presuppositions of the inference method are true.
$\therefore$ C. For any (most?) random procedures, all events $E \subseteq \Omega$ have a chance.

Objection 5 Totality is inconsistent with the rotational/translational symmetry of the chance measure. Reply. Yes, but the conclusion to draw is that spinning tops, roulette wheels etc. have rotationally asymmetric chance distributions, not that they are partial. Quite independently of this incompatibility, we have every reason to think that these physical objects are never perfectly symmetric.

Lebesgue (1904). There is a chance measure Ch on the continuous circle $\mathbf{C}$ such that:
B. Zero-Chance Outcomes
D. Countable Additivity
E. Lebesgue Totality. The chance $\operatorname{Ch}(L)$ is defined for all and only Lebesgue subsets $L \subseteq \mathbf{C}$.
F. Lebesgue Symmetry. For every Lebesgue subset $\mathrm{L} \subseteq \mathrm{C}$ and every rotation $\rho, \operatorname{Ch}(\rho L)=\operatorname{Ch}(L)$.

Vitali (1905). Assuming ZFC, there is no chance measure Ch on the continuous circle C such that:
B. Zero-Chance Outcomes
D. Countable Additivity
C. Totality
G. Total Symmetry. For every subset $S \subseteq \mathbf{C}$ and every rotation $\rho, \mathrm{Ch}(\rho S)=\mathrm{Ch}(S)$

Solovay (1970). It is consistent with ZF without Choice that all subsets of the continuum are Lebesgue (assuming ZF + "There is a strongly inaccessible cardinal" is consistent).

Objection 6. Whenever physicists or applied mathematicians define a chance measure on a continuum $\Omega$, it is always a partial measure. Doesn't that show they are not presupposing the chance measure to be total?
Reply. It just shows that this presupposition is not made explicit in the chance measures they use. The practice of using partial chance functions is practically well-motivated independently of whether you think there are chance-free events. By ignoring non-Lebesgue events, one gains a valuable symmetry, making the maths much simpler. But it implies no view about the status of the sets one is ignoring.

## D. Countable Additivity

D1. The mathematical techniques scientists routinely use to model chances in a situation with a Continuum of Outcomes have been instrumental in securing many empirical successes.
D2. These techniques include integration of probability densities and laws of large numbers, which presuppose countably additive chance measures.
D3. The best explanation for D1 is that the assumptions these techniques rely on are true.
$\therefore$ D. Chance measures are countably additive.

Objection 6. Suppose I know the bell will toll exactly once in an infinite future, and that is all I know. Then the likelihood the bell will toll tomorrow is is equal to the likelihood that the bell will toll $n$ days from now, for any $n$. Consequently the likelihood of "the bell will toll on day $n$ " cannot be positive for any $n$. And yet the likelihood that the bell will toll some day is 1 , in violation of countable additivity.
Reply. This is an argument against countably additive credences. There is no analogous argument reason to doubt the countable additivity of chances. For instance, there is no physical procedure for fairly selecting a random natural number.

Note. In my paper I put forward another argument for countably additive chances, arguing that countably non-additive chance distributions cannot be confirmed, and that it is irrational to give them any credence. This argument reinforces the disanalogy between chance and credence: the positive case for countably additive chance does not extend to credences, either.

## Banach and Kuratowski 1929

Definition. Let $\mathbf{N}=\{0,1,2, \ldots\}$ be the set of all natural numbers. Then if $f$ and $g$ are functions from $\mathbf{N}$ to $\mathbf{N}$, let us say $f$ is bigger than $g$, written $f>g$, just in case $f(n)>g(n)$ for all but finitely many natural numbers $n \in \mathbf{N}$. (It is easily checked that $>$ is a strict partial order.)

Lemma. If $2^{\aleph_{0}}=\aleph_{1}$, then there is a continuum-sized set $\Omega$ of functions from $\mathbf{N}$ to $\mathbf{N}$ such that for any function $g$ from $\mathbf{N}$ to $\mathbf{N}$, all but countably many members of $\Omega$ are bigger than $g$.
Proof. The cardinality of the set of all functions from $\mathbf{N}$ to $\mathbf{N}$ is $2^{\aleph_{0}}$, so if $2^{\aleph_{0}}=\aleph_{1}$, we can let $\left\{g_{\alpha}: \alpha \in \aleph_{1}\right\}$ be an enumeration of all those functions by the countable ordinals. Now we can construct the desired set $\Omega=\left\{f_{\alpha}: \alpha \in \mathcal{N}_{1}\right\}$ with the following recursion:
$f_{0} \quad: n \mapsto 1$
$f_{\alpha+1}: n \mapsto f_{\alpha}(n)+g_{\alpha}(n)$
$f_{\lambda} \quad: n \mapsto \sum_{k=0}^{n} f_{\beta_{k}}(n) \quad$ where $\left\{\beta_{k}: k \in \mathbf{N}\right\}$ is any enumeration of the ordinals below $\lambda$
Note that for any function $g_{\alpha}, g_{\alpha}<f_{\alpha+1}$. And since the sequence $\left\{f_{\alpha}: \alpha \in \mathcal{N}_{1}\right\}$ is increasing under $<$, it follows $g_{\alpha}<f_{\beta}$ for all countable ordinals $\beta>\alpha$, which is to say for all but countably many $f \in \Omega$.

Theorem. If a chance measure Ch satisfying conditions (A-D) exists, then $2^{\aleph_{0}} \neq \aleph_{1}$.
Proof. Let $\Omega$ be any continuum-sized set of functions from $\mathbf{N}$ to $\mathbf{N}$. Let Ch be a chance measure on $\Omega$. For any $n$, consider the partition $P_{n}$ of $\Omega$ that sorts the functions in $\Omega$ according to their value for $n$ :

$$
P_{n}(k)=\{f \in \Omega: f(n)=k\}
$$

Now since $\operatorname{Ch}\left(\cup_{k=0}^{\infty} P_{n}(k)\right)=\operatorname{Ch}(\Omega)=1$, by Countable Additivity it follows that $\sum_{k=0}^{\infty} P_{n}(k)=1$, whence there must be some $k_{n}$ such that

$$
P_{n}(0)+P_{n}(1)+\ldots+P_{n}\left(k_{n}\right)>1-1 / 2^{n+2}
$$

Letting $S_{n}=P_{n}(0) \cup P_{n}(1) \cup \ldots \cup P_{n}\left(k_{n}\right)$, by finite additivity $\mathrm{Ch}\left(S_{n}\right)>1-1 / 2^{n+2}$. Now let $S=\bigcap_{n=0}^{\infty} S_{n}$. Using Countable Additivity, $\mathrm{Ch}(S)>1-\sum_{n=0}^{\infty} 1 / 2^{n+2}=1-1 / 2=1 / 2$. So $S$ must be uncountable (else $\mathrm{Ch}(S)=0$ by Countable Additivity and Zero-Chance Outcomes). Now note that for any $f \in S$, and any $n \in \mathbf{N}, f \in S_{n}$, whence by definition $f \in P_{n}(k)$ for some $k \leq k_{n}$. But that just means $f(n) \leq k_{n}$. Thus for all $f \in S, f<g_{\Omega}$ if $g_{\Omega}: n \mapsto k_{n}+1$. Thus for any $\Omega$, once can construct a function $g_{\Omega}$ such that $f<g_{\Omega}$ for uncountably many $f \in \Omega$. From the lemma above it follows $2^{\aleph_{0}} \neq \mathcal{\aleph}_{1}$.

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