

Logical Subtraction (Except Maybe Not)

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Suppose I say that P and add that Q . Then I have said that $(P \wedge Q)$. Call that logical *addition*:

$$\top + P = P$$

$$P + Q = (P \wedge Q)$$

So logical addition is basically just conjunction.¹ Now suppose I start by saying $(P \wedge Q)$ and then take back Q . Then I might be off the hook for Q , but I'm still on the record for P . If I then go on to take back P as well, I am effectively back to having no commitments. We might call that logical *subtraction*:

$$(P \wedge Q) - Q = P$$

$$P - P = \top$$

Logical subtraction is something like the inverse of conjunction.

While conjunction is a familiar, unproblematic and well-understood, its inverse, logical subtraction, is steeped in mystery. As long as we restrict ourselves subtracting conjuncts from their conjunctions, no particularly interesting questions arise. But what happens when subtracting other entailments?

I. Explicit Subtraction

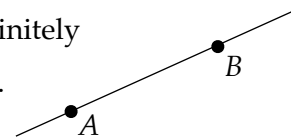
In English, you can often take part of what you said back by using the word “except”. More specifically the phrase “except maybe not” seems like a promising candidate for a natural language subtraction operator. That is to say, the expression “ P , except maybe not Q ” often expresses the message $P - Q$:

1. It was a rainy Wednesday in New Jersey, except that maybe it was not raining, maybe it was not Wednesday and maybe it was not in New Jersey.
2. An autocracy is like an absolutist monarchy, except that the ruler need not be a king or queen.
3. The store is open every day. Except perhaps on Sundays.
4. Sarah is very fair-minded. Except maybe when it comes to her children.
5. Pescatarians are vegetarians, except that they might eat fish.

¹ In the Boolean tradition conjunction is often analogised to *multiplication* rather than addition (and disjunction is thought of as analogous to addition). This is a natural analogy when you are thinking about the truth values of the sentence letters as standing in for zeroes and ones. But when thinking instead about the propositional contents of the sentences, the analogy between conjunction and addition is more natural: when I say “Swedes are tall and they are friendly,” I have not in any sense *multiplied* the claim that *Swedes are tall*. But I have *added* something to it. See discussion of Peirce below.

In some cases, Q is not merely subtracted but “strongly retracted”, by which I mean that it is replaced by a claim incompatible with Q , often just $\neg Q$. Here are some examples of such strong retractions, which might be formalised as $(P - Q) \wedge \neg Q$ or $(P - Q) \wedge R$:

6. A gratin is a quiche. Except it is not baked in a shell. (Fuhrmann 1999)
7. A hypothesis is a belief, except it need not be seriously accepted. (Fuhrmann 1999)
8. Our customers can have a car painted any colour they want. Except if it’s not black. (Henry Ford)
9. The rules of field hockey and ice hockey are exactly the same. Except there is no ice.
10. This is a picture of the line AB . Except that Euclidean lines are infinitely thin, with no width at all, and continue forever in both directions.



Philosophy is full of examples of (attempted) subtractions.

“All ratiocination is comprehended in these two operations of the mind, addition and subtraction ... [When] the idea of rational is subtracted from the whole idea of man, that is to say, of body-animated-rational, there remains that of body-animated ... We must not therefore think that computation, that is ratiocination, has place only in numbers ... for [all the kinds of philosophy] are capable of addition and subtraction.”

— Hobbes, *De Corpore*, 1655 (EW i.3)

“When ‘I raise my arm,’ my arm goes up. And the problem arises: what is left over if I subtract the fact that my arm goes up from the fact that I raise my arm?”

— Wittgenstein, *Philosophical Investigations*, §62

“In action, world is adapted to mind. In knowledge, mind is adapted to world. When world is maladapted to mind, there is a residue of desire. When mind is maladapted to world, there is a residue of belief. Desire aspires to action; belief aspires to knowledge.”

— Williamson, *Knowledge and Its Limits*

11. A statement is *lawlike* if it is a law, except it might not be true. (Goodman 1983)
12. A statement has *warrant* if it is known, except that it might not be believed. (Plantinga 1993)
13. *Lying* is asserting an intentional falsehood, except that it may be true by accident. (Grotius)
14. Macbeth *sees** a red dagger just in case he sees a red dagger, except there might be no red dagger there for him to see. (Ayer 1940)
15. There is a differentiable function that maps from the space-time manifold to the real numbers, except that there are no mathematical objects. (Melia 2000/Colyvan 2010)
16. Oscar thinks there is water in his glass, except that he may never have encountered water, for instance because he lives on twin earth. (Loar 1987)

II. Subtraction Failures

Sometimes attempts at subtraction can fail. I have gathered some examples below. To bring out the clash in these cases to maximal effect, I have stated them in the “strong retraction” form $(P - Q) \wedge \neg Q$. But for the most part, these still sound bad after you insert a “maybe”.

17. At the wedding, Holly played the violin very badly. Except that she did not play the violin.
18. There are exactly seven planets in our solar system. Except that the number of planets is not a prime.
19. The patch is red. Except that it is not coloured. (Jackson 1977)
20. The tomato is scarlet. Except that it is not red [but blue!]. (Woods 1967)
21. Jill was born in Vietnam. Except that she was not born in Asia [but in Europe!].
22. It snowed. Except that it may not have been the case that either it snowed or it rained.
23. There are six footprints in the sand. Except the impressions are may not be caused by feet. (Stalnaker 1989)
24. I swam three laps. Except that I didn't swim. (Yablo 2012)
25. Fred is thirsty. Except that no-one is thirsty. (Yablo 2012)
26. When Mr. Bilbo Baggins of Bag End announced that he would shortly be celebrating his eleventy-first birthday with a party of special magnificence, there was much talk and excitement in Hobbiton “Well, I'm back.” he said. And that's how it happened, except without all the hobbits. (Colyvan 2010)
27. The wireless telegraph works exactly the same way as the regular telegraph. Except without all the wires. (Yablo 2014, §8.1)

These cases make it clear that we should probably not expect $P - Q$ to be defined for every choice of Q . That means a systematic account of subtraction should say something about when we can and cannot subtract a proposition from another. A natural first restriction would be to say that Q has to be a *consequence* of P , or part of P . But that may not be enough. According to Yablo for instance, we do not get a well-defined remainder $P - Q$ when trying to subtract a part Q that is *inextricable* from P . (It may also be too much: arguably, we can often make sense of $P - Q$ even when Q is not entailed by P .)

A more pessimistic reading of these examples is to conclude that it is impossible to make systematic sense of the notion of logical subtraction. Until recently at least, that was the received view amongst logicians who had considered the matter. Hence the title of this handout: this will be a class about logical subtraction, except that there may not be any such subject matter.

III. Overview of the Course

Over the course of the next class, we will build a toolkit, introducing some semantic concepts that will be helpful on our exploration. After that, we will examine a particular philosophical application of logical subtraction every week, paired with a theory of what subtraction is. Except for the last week, when there will be no pairing.

▶ **Week 3. Loose Talk and Metaphors**

Gist: Italy is a boot, except not exactly a boot.

Pairing: Yablo's *Aboutness* and my *Conversational Exculpature*.

Slogan: Subtraction is restoring relevance.

▶ **Week 4. Nominalism / Fictionalism about Maths**

Gist: The number of people in this class keeps increasing, except there are no numbers.

Pairing: Lloyd Humberstone, *Parts and Partitions*

Slogan: Tell me how to divide the world, and I will tell you how to divide propositions.

▶ **Week 5. Other Kinds of Anti-Realism**

Gist: The supermarket is around the corner, except there are no objects with proper parts.

Pairing: Kit Fine, *A Theory of Truthmaker Content*.

Slogan: Logical subtraction, except that it may not be the inverse of conjunction.

▶ **Week 6. Frege's Puzzle**

Gist: Lois believes Clark cannot fly and she believes that Superman can, except that Superman and Clark are in fact the same person.

Slogan: Leibniz' Law is overrated.

Though the literature on subtraction is very small, we will not quite cover all of it. Let me mention some relevant texts that we will not cover:

- ▶ Lloyd Humberstone's extensive but ultimately inconclusive investigations into the concept of logical subtraction in Humberstone 1981 and 2011.
- ▶ Bas van Fraassen, Ken Gemes' and Richard Angell's work on parthood and analytic or tautological entailment (Van Fraassen 1969, Gemes 1994, 1997, Angell 1977).
- ▶ André Fuhrmann's account in terms of hyperpropositions (Fuhrmann 1996, 1999).
- ▶ The literature on belief revision. An agent who revises their belief does not subtract one proposition from another, but rather subtracts a proposition from a body of information. This is an extensive literature: see e.g. Gärdenfors 1988, Van Ditmarsch, Van der Hoek, and Kooi 2007.

This week, we will look at two simple early theories of subtraction. One is that of C.S. Peirce (1867). The other is Hudson’s (1975) material conditional analysis of subtraction, proposed in response to Jaeger. Other discussions of subtraction in the specific context of Wittgenstein’s thoughts on arm-raising include Jaeger’s (1976) retort to Hudson, Vesey 1966, Hornsby 1980 and Tanney 2018.

IV. Jaeger’s Problem

Hudson starts from the properties for subtraction listed by Jaeger. If $P - Q = R$, then

- I. P entails R (“What is left over will be a part of the original whole.”)
- II. $Q \wedge R$ entail P (“The whole is equal to the sum of its parts.”)
- III. R does not entail Q (“What is subtracted cannot be a part of what is left over.”)
- IV. Q does not entail R (“What is left over cannot be part of what was subtracted.”)

Jaeger also takes the following background conditions for granted:

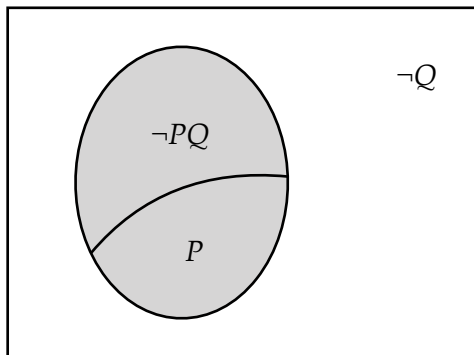
- O. P strictly entails Q , and Q is no tautology (“A proper part of the whole is subtracted, and the part subtracted is not null.”)

Now for Jaeger, the problem is this: for given P and Q , there is no *unique* proposition R that satisfies conditions (I-IV). For example, both $(Q \supset P)$ and $(Q \wedge S) \supset P$ satisfy (I-IV), for arbitrary choice of S . More concretely, by the lights of I-IV the following both seem like equally good candidates for *I raise my arm – My arm goes up*:

- ▶ $(My\ arm\ goes\ up \supset I\ raise\ my\ arm)$
- ▶ $(My\ arm\ goes\ up \wedge Aristotle\ admired\ Plato) \supset I\ raise\ my\ arm$

Of course neither are particularly good candidate analyses for Wittgenstein’s intended target: *I will my arm to go up*.

We can use the following diagram to map out various possibilities for R , where the points represent possible worlds, and the regions propositions (areas of logical space); in particular, the shaded region represents the subtracted proposition Q :



Hudson's proposal is to add an additional constraint.

- V. Every proposition that satisfies (I-II) entails R ("What is left over is the smallest part of P that bridges the gap between the whole and what was subtracted.")

Now it turns out that conditions (I-II) and (V) jointly pick out a unique proposition (up to truth conditions), and do so independently of whether (O) is satisfied. That proposition is $(Q \supset P)$. Hence Hudson's proposal:

$$P - Q = Q \supset P$$

This view has a lot going for it. In particular, this remainder for the most part satisfies Jaeger's conditions (III) and (IV): whenever (O) is satisfied they are satisfied, and even the exceptions are intuitively well-motivated.² Still, this notion cannot do the theoretical work we wanted logical subtraction to do. (Why not?)

C.S. Peirce's (1867) brief discussion of subtraction takes a different tack, and gives up the demand for uniqueness. He specifies the range of possible remainders as follows:

$$P - Q = \begin{cases} \text{undefined} & \text{if } P \text{ is consistent with } \neg Q \\ (P \wedge Q) \vee (V \wedge \neg P \wedge \neg Q) & \text{otherwise} \end{cases}$$

Where V is allowed to be any proposition. Again, this is a principled solution to the problem Jaeger poses, but not one that can bear the theoretical load we want a notion of subtraction to carry: if we want subtraction to play a role in philosophical analysis, we cannot allow this level of ambiguity.

(Actually, Peirce expresses all this much more elegantly than I just did. Since he is working in a Boolean framework, he is thinking of conjunction as *multiplication* rather than addition, and so what we have been calling subtraction, he thinks of as division — subtraction would be the inverse of *disjunction*. This allows a particularly neat characterisation of division:

$$\frac{P}{Q} = P \cdot Q + v \cdot \bar{P} \cdot \bar{Q} + \frac{P \cdot \bar{Q}}{0}$$

If $P \cdot \bar{Q} \neq 0$, this is undefined; if $P \cdot \bar{Q} = 0$, the zeroes cancel each other out; Peirce 1867, p. 15.)

² If $Q = \top$, obviously (III) is violated; (IV) is violated if and only if Q entails P . Both these exceptions to (III) and (IV) are consequences of the fact that a tautology is entailed by every proposition, just like the null part is part of everything. Since the null part is part of everything, it is bound to be part of the remainder when subtracted. And if you subtract everything, only the null part is left, which will be then be part of what was subtracted. These are not violations of the basic intuition that is motivated Jaeger in positing (III) and (IV), which is that Q and R should be disjoint.

Yablo 2006 also considers the following conditional analyses of subtraction:

$$P - Q = Q \rightarrow P$$

$$P - Q = Q \Box \rightarrow P$$

Where “ \rightarrow ” and “ $\Box \rightarrow$ ” stand for the indicative and counterfactual conditional respectively. But he quickly notes that these analyses runs into trouble pretty quickly. In particular, both (28) and (29) are most naturally interpreted as meaning very different things than (30).

- 28. If Joe raised his hand, everyone did.
- 29. If Joe had raised his hand, everyone would have.
- 30. Everyone raised their hand. Except maybe Joe.

Nonetheless, Yablo 2016 explores the relation between indicative conditionals and subtraction, as does Khoo 2013.

V. Implicit Subtraction

31. Definite Descriptions.

- a. The man drinking a martini is a notorious jewel thief. (Donnellan)
- b. The Waynflete Professor of Logic is older than I am (Strawson)
- c. The lodger next door has offered me twice that sum (Strawson)
- d. I had breakfast with the King of France this morning (Yablo)
- e. The dagger Macbeth saw in front of him was pointed towards his hand. (Lewis)

32. Fictional Characters and other Non-Existent Entities

- a. Harry Potter did not go to not Hufflepuff, he went to Gryffindor! (Recanati)
- b. Eli is wearing a hat just like the one Holmes always wears.
- c. Hob thinks a witch burned down his barn, Nob believes she blighted his mare. (Geach)
- d. Joe wished the man who robbed him had never robbed anyone. (Blumberg)

33. Waltonian Metaphors

- a. Crotone is in the arch of the Italian boot. (Walton)
- b. Look at that thundercloud over there! The big angry face behind the oak tree. (Walton)
- c. I watched her drift slowly out to sea, until she became a dot on the horizon. (Kripke)
- d. The weather gods have been kind to us lately.

34. Loose talk

- a. Joe arrived at six fifteen.
- b. There were six thousand people at the rally.
- c. Over the last five years, the library lent out over a million books. (Krifka)
- d. France is hexagonal. (Lasersohn 1999)

35. Idealisation

- a. The volume of the fridge is length times height times width.
- b. Electrons are point particles.
- c. In this balloon of helium, $PV = nRT$.
- d. In a given situation, people do whatever best promotes their desires given their beliefs.

VI. Sneak Preview

- ▶ A kind of entailment called *propositional parthood*. It is natural to think that in subtraction we subtract a *part* of the original claim, and that the remainder should itself also be a part.
 - ▶ Conjuncts are parts of their conjunction, but disjunctions are not part of their disjuncts.
 - ▶ Partial Truth as Truth of a Part.
 - ▶ Part of what is said.
- ▶ *Subject matters* (Lewis 1988). *The moon* is a subject matter. Other subject matters include *cars*, *the nineteenth century* and *how many stars there are*. Having a precise conception of subject matters will help us capture the thought that in subtraction, we subtract what *P* says about one subject matter and retain what it says about some other subject matter:

“There is no reason why we should not introduce some new expression, such as “So far as he, but not necessarily his arm, was concerned, he moved his arm,” or “So far as the mental side of him as an agent is concerned, he moved his arm,” at the same time stipulating that it is to work like “He moved his arm,” except that it can be true even when ‘His arm moved’ is false.” (Vesey 1964)

- ▶ Subject matters can be part of one another, they can be fused and they can overlap.
 - ▶ For instance, *1879* is part of *The Nineteenth Century*, *oak trees* is part of *trees* and *my arm* is part of *my body*. In each case, knowing absolutely everything there is to know about the latter, means knowing everything there is to know about the former.
 - ▶ Upwards difference transmission: you can’t change something about 1879 while leaving the nineteenth century unaffected.
 - ▶ What is the overlap between *The capitals of Europe* and *The capitals of Asia*?
- ▶ Subject matters as partitions.
- ▶ *Truthmakers* (Yablovian and Finean). These allow us to make fine distinctions in meaning that are potentially important in this context.
 - ▶ Truthmakers are ways of being true, and falsemakers ways of being false.
 - ▶ Truthmakers are situations, states or small possible worlds.
 - ▶ Hyperintensionality

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