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Mathematics and Metaphor

Definitions:

a) A *proposition* is (represented by) a set of possible worlds. A proposition is true at all and only those worlds that are its members, and false everywhere else.

b) A *subject matter* U is a partition of logical space (Ω). That is, a subject matter is a set of non-empty, disjoint sets of possible worlds, whose union is Ω. We will refer to these disjoint sets as the *cells* of the subject matter. (We write $w \sim_U v$ if and only if w and v are in the same cell of a subject matter U.)

c) A proposition p is *about* U if and only if $p = \cup X$ for some set of cells $X \subseteq U$.

d) A proposition p has *no bearing* on a subject matter U if and only if $\varnothing \neq p \cap [u] \neq [u]$ for all cells $[u] \in U$.

e) A *partial proposition* is (represented by) an ordered pair $\langle a, b\rangle$, where a and b are sets of worlds and $a \subseteq b$. $\langle a, b\rangle$ is true at w iff $w \in a$, and false at w iff $w \in b\setminus a$. The truth value of $\langle a, b\rangle$ is undefined outside b. (The pair $\langle a, \Omega\rangle$ represents a full proposition, viz. the same full proposition as the set a.)

f) The *restriction* of p to q, written $p{\upharpoonleft}q$, is the partial proposition $\langle p \cap q, q\rangle$.

g) The *partial proposition* $\langle a, b\rangle$ is *about* the subject matter U if and only if $\langle a, b\rangle = r{\upharpoonleft}b$ for some full proposition r about U.

h) Suppose $\langle a, b\rangle$ is a partial proposition about U. Then the *completion* of $\langle a, b\rangle$ by U, written $U(\langle a, b\rangle)$, is the following (partial) proposition:

$$U(\langle a, b\rangle) =_{\mathrm{df}} \langle\{w : w \sim_U v \text{ for some } v \in a\}, \{w : w \sim_U v \text{ for some } v \in b\}\rangle$$

The Useful result:

Let r, p, q be full propositions and let U be any subject matter. Then $r = U(p{\upharpoonleft}q)$ if and only if the following three conditions obtain:

- r is about U
- $p{\upharpoonleft}q = r{\upharpoonleft}q$ (that is, conditional on q, p and r are equivalent)
- q entails nothing substantial about U (that is, q entails no proposition about U other than the necessary truth)

In case only this final condition fails, we have $U(p{\upharpoonleft}q) = r{\upharpoonleft}s$, where s is the strongest proposition about U entailed by q.

[EXPLANATION: The first condition holds iff r, just like $U(p{\upharpoonleft}q)$, has only one truth value per cell of U (def. (c)). Given that this is the case, the second condition holds iff those truth values are the same with respect to cells that are compatible with q. In order to entail something substantial about U, q would have to rule out one of the U cells. So q entails nothing substantial about U iff it is compatible with every cell in U. So the third condition holds iff $\Omega = \{w : w \sim_U v \text{ for some } v \in q\}$, i.e. iff $U(p{\upharpoonleft}q)$ is everywhere defined (def. (h)).]

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The Myth of the Natural Numbers

'Beyond the outer reaches of our physical universe and outside our temporal dimension, is the eternal realm of the Platonic Forms. Amongst its inhabitants are the proud and unchanging Natural Numbers, who sit enthroned on the natural number line.

On the leftmost side of the line, on a throne made of baseballs, sits the number Zero. Zero is the number belonging to the class of natural numbers sitting to the left of Zero, and to all and only classes equinumerous to that class. Immediately to her right, on a throne made of matchsticks, sits One. One is the number belonging to the class of natural numbers sitting to the left of One, and to all and only classes equinumerous to that class. Immediately to the right of One, on a throne made of white feathers, sits the number Two. Two is the number belonging to the class of natural numbers to the left of Two, and to all and only classes equinumerous to that class. Etcetera, etcetera.

For every natural number, there is another natural number seated on its immediate right. Every natural number belongs to the class of natural numbers seated to its left and to all and only classes equinumerous to that class. For every natural number, the class of natural numbers on its left finite.'

[NOTE: This myth is formulated in a second-order language. For our immediate purposes, the following very simple theory of classes does the job: 1. There is a class containing nothing; 2. For any object x, there is a class containing x and nothing else; 3. For any two classes A and B there is a class containing all and only the objects contained in A and B. (We don't even need extensionality). Depending on whether you are a nominalist about classes, you can write these classes into the platonic myth, or assume they are already part of the nominalistic universe. A class A is *finite* iff any injective function from A to A is also surjective. Two classes A and B are *equinumerous* iff there is a bijection between the objects in A and the objects in B.]

"Conservativity"

For any propositions p_i, $i \in I$, q and c such that $U(p_i \upharpoonright q)$ and $U(c \upharpoonright q)$ are all well-defined, the following holds:

If $\{p_i : i \in I\} \vDash c$, then $\{U(p_i \upharpoonright q) : i \in I\} \vDash U(c \upharpoonright q)$

Proof: Without loss of generality, take the set of all worlds to be $\{w : w \sim_U v$ for some $v \in q\}$, so that $U(p_i \upharpoonright q)$ and $U(c \upharpoonright q)$ are total. We need to show that $\{U(p_i \upharpoonright q) : i \in I\} \vDash U(c \upharpoonright q)$. So we need to show $\cap_i U(p_i \upharpoonright q) \subseteq U(c \upharpoonright q)$. We divide this proof into two parts: first, we'll show that the entailment holds w.r.t. q-worlds:

1. $(\cap_i U(p_i \upharpoonright q)) \cap q \subseteq U(c \upharpoonright q) \cap q$

Note that $U(x \upharpoonright q)$ always matches x in q-worlds, so that the intersections of $U(x \upharpoonright q)$ and x with q are identical. Thus, using the fact that $(\cap_i U(p_i \upharpoonright q)) \cap q = \cap_i (U(p_i \upharpoonright q) \cap q)$, (1) simply reduces to

2. $(\cap_i p_i) \cap q \subseteq c \cap q$

And (2) in turn follows from $\cap_i p_i \subseteq c$, which is given.

Now let w be any world in $\cap_i U(p_i \upharpoonright q)$. We assumed w, like all worlds, is such that $w \sim_U v$ for some $v \in q$. Since $U(p_i \upharpoonright q)$ is about U, it follows from the fact that $w \in U(p_i \upharpoonright q)$ that $v \in U(p_i \upharpoonright q)$ for all i. Hence $v \in U(p_i \upharpoonright q) \cap q$. Thus $v \in (\cap_i U(p_i \upharpoonright q)) \cap q \subseteq U(c \upharpoonright q) \cap q$, from which it follows that $v, w \in U(c \upharpoonright q)$. ▩