In Defence of the Romance of Mathematics

Or: How Should Cognitive Scientists and Philosophers Talk to One Another About Maths?

Different Interests

Central Questions in the Cognitive Science of Mathematics

- **Cognition:** How do we think about mathematics? What is the role of mathematics in our broader cognitive lives?
- **Acquisition**: How are mathematical concepts acquired? What are the (biological) conditions under which they can be acquired?
- **Didactics**: What are effective methods for teaching mathematics?
- **Practice**: What are the cognitive and heuristic roles of mathematical proofs?

Central Questions in Traditional Philosophy of Mathematics

- **Ontology**: Do mathematical objects like numbers really exist?
- **Metaphysics**: What is the nature of mathematical objects? What is mathematical truth?
- **Epistemology**: What is the justification for our mathematical beliefs? How is mathematical knowledge possible?
- **Application**: How does mathematics apply to the physical world? Why is it mathematics plays such a central role in the other sciences?

Status quo: Currently each side investigates its own problems in isolation from the other. Where issues from the other side come up (as they inevitably do), investigators hastily assume uninformed answers to those questions, showing little awareness that dedicated studies on these matters exist.

Proposal: We should understand the differences between the questions that guide our different fields to avoid merely apparent conflicts. At the same time, it is worthwhile to strive for better communication. That way, we can each continue to pursue our own lines of inquiry with

A) knowledge of the results and insights acquired on the other side; and
B) clarity about the boundaries of our respective investigations

This should benefit both enterprises. As an added benefit, better communication opens the gates to more interdisciplinary work.

The "Romance of Mathematics"

Lakoff and Núñez (2000) identify the following theses (amongst others) as forming part of what they call the *Romance of Mathematics*, a common set of beliefs about the discipline:

- **Abstraction:** Mathematics is (or mathematical objects are) abstract and disembodied.
- **Existence:** Mathematics has an objective existence, independent of and transcending the existence of human beings or any beings at all.
- **Objectivity:** What human beings believe about mathematics therefore has no effect on what mathematics really is.
- **Necessity:** Mathematicians discover absolute truths not just about this physical universe but about any possible universe.
- **Scrutability:** Mathematical proof allows us to discover transcendent truths of the universe.
- **Indispensability:** "The book of nature is written in mathematics" which implies that the language of mathematics is the language of nature and that only those who know mathematics can truly understand nature.
- **A Priorism**: Mathematics is the product of pure reason, unadulterated by experience.

While these statements are a bit ambiguous, most of these claims, in one form or another, have the status of orthodoxy in philosophy of maths. Lakoff and Núñez argue they are all shown to be false, unscientific and baseless by their observations on mathematical cognition. In making their rather quick and somewhat naïve arguments, they commit two sins that present an obstacle to rapprochement between cogsci and philosophy:

- They show themselves unaware of the strong arguments for these positions, and the problematic consequences that attach to their rejection. This unawareness also shows in the fact that they are not altogether consistent in their rejection of the romance.
- They conflate the questions of cognitive science with those of philosophy, and consequently misstate the import of their insights. They even coin a slogan for this error: "mathematics *is* human mathematics".

Mathematics as a Field of Study vs. Mathematics as an Object of Study

"Botany *is* human botany." That's plausible enough: to argue that this is true in a sense, one only needs to point out that other creatures (say, grasshoppers or octopuses) do not seem to engage in activities that could be classified as botany. All the botany that is done at all is done by humans (as far as we know). But no-one would or should conclude from this that *plants* are human mental constructs, without an "objective, external existence". Similarly one must not slide from the observation that the *study* of mathematics is a peculiarly human activity to the conclusion that its *object* (also called "mathematics") is some mental construct. (E.g. what about Gödel?)

Case Study: Mathematics as a Metaphor

Fictionalism about Mathematics (Field 1980/2016)

Thesis: *Mathematical objects like numbers and functions do not really exist; they have been made up. Thus mathematical objects have much the same status as fictional objects or characters, and mathematical statements have much the same character as fictional statements.*

Consequently, the following statements are literally false:

1) "Seven plus five equals twelve"
2) "Eight minus three equals thirteen"
3) "The number of planets in our solar system is greater than two"
4) "There are infinitely many primes"

while the following statements are literally true:

5) "It is not the case that eight minus three equals thirteen"
6) "There are no primes between ten and twenty"
7) "No kiwi has an even number of seeds"
8) "There are no positive integer solutions to the equation $a^n + b^n = c^n$ with $n > 2$"

Conversational Exculpature (my view; cf. also Yablo 2014)

Theory (simplified): Suppose in a conversation with subject matter/question under discussion S, a speaker makes an utterance with literal content p while contextually presupposing q. Suppose further that there is a third proposition r that uniquely satisfies the following two conditions:

A) Conditional on q, p and r are equivalent
B) r is wholly relevant to the subject matter S

Then r is available as a non-literal reading of the utterance.[1]

Putative examples:

p_1: *Ellen owns a hat of the same model as the one Sherlock Holmes always wears*
q_1: *Holmes always wears a deerstalker*
S_1: *Ellen's hat collection*
r_1: *Ellen owns a deerstalker*

p_2: *The weather gods are fickle these days.*
q_2: *The weather is completely controlled by weather gods*
S_2: *The recent weather*
r_2: *The recent weather is changeable*

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[1] In the formal implementation of the theory, *propositions* (like p, q and r) are subsets of logical space, and *questions* or *subject matters* are partitions of logical space. A proposition is p is *wholly relevant* to a question S just in case p is a union of S-cells.

Mathematical Exculpature

If fictionalism about mathematics is true, there is a discrepancy between the literal contents of statements like (1-8) and the messages they intuitively communicate (for one, they have different truth values). The theory of exculpature can explain this discrepancy.

For any proposition *p*, let $\circlearrowleft p$ denote the result after exculpature given the following contextual parameters:

- **Contextual presupposition**: *Beyond the outer reaches of our physical universe, there is the Platonic Realm of Mathematics. Amongst the denizens of this enchanted land are the unchanging Natural Numbers, arranged on the eponymous Natural Number Line. All the way on the left sits the number Zero. Immediately to Zero's right sits One. To the right One sits Two, etcetera. To the immediate right of every natural number sits another natural number. Every natural number numbers the class of natural numbers seated to its left and all and only classes equinumerous to that class. The End*
- **Subject matter**: *The concrete world*

Intuitively, the operator "$\circlearrowleft$" takes in propositions that make reference to natural numbers, and spits out their concrete upshot, a proposition that makes no reference to abstracta:

p	$\circlearrowleft$***p***
The number of planets in our solar system is greater than two	*There is a planet in our solar system, and another, and another.*
No kiwi has an even number of seeds	*Every kiwi is such that its seeds can be divided up into two groups where the seeds in one group can be paired up 1-1 with seeds in the other*
The number of giraffes is greater than the number off elephants	*There is a group of giraffes such that all the elephants in the world can be paired up 1-1 with the giraffes in the group. There are also giraffes that are not in this group.*
Seven plus five equals twelve	$\top$ (the necessary truth)
Eight minus three equals thirteen	$\bot$ (the necessary falsehood)
There are no positive integer solutions to the equation $a^n + b^n = c^n$ *with* $n > 2$	$\top$
There are strongly inaccessible cardinals	undefined!
…	…

Mathematics as a metaphor: When using mathematics, speakers are always contextually presupposing a certain mathematical myth. The non-literal message they thereby express is the result of conversationally exculpating that myth from the literal content of their utterance.

Morals

The thesis labeled "Mathematics as a metaphor" is a view in the philosophy of mathematics, targeted at answering the traditional questions mentioned at the outset. It also underwrites most aspects of the "romance of mathematics". Yet at the same time it is clearly very harmonious with Lakoff and Núñez' observations on mathematical cognition. In particular:

- It acknowledges (and makes precise) the relation between mathematics and other metaphors.
- It makes sense of the existence of distinct number metaphors: variations in the contextual presupposition can be tolerated as long as they don't affect the outcome of the exculpature. E.g. the version of the myth given above uses the line metaphor, but a sets-of-units version of the story would give exactly the same results.
- The existence of this multiplicity is useful in that different stories admit of different extensions ("sequels"). E.g. the line story can be extended to include negative and rational numbers, while the sets of units story can be extended to include infinite cardinalities.
- It underwrites the thesis that mathematical objects are in some sense human creations (in the same way that Sherlock Holmes is the creation of Conan Doyle).

Thus the existence of this particular view illustrates that the philosophical conclusions Lakoff and Núñez draw from their psychological theory do not in fact follow (cf. also Frege's 1884 critiques of psychological theories of mathematics). They create the appearance of a deep conflict between cognitive science and philosophy where there is in fact none.

The view of mathematics as a metaphor can be motivated by insights from cognitive science. At the same time it has the potential to inform the development of theories of mathematical cognition. Thus it opens prospects for the collaborative development of comprehensive views of mathematics that harmonises with the perspectives and demands of all applicable fields. Other philosophical views, like structuralism, may admit of similarly cogsci-friendly implementations.

References

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