What lies beyond

Congratulations! Now you know logic. Hopefully you found the subject to be of interest in itself. But whether or not you did, you ought to be aware that logic is not just there for its own sake. It is a tremendously versatile tool, that opens the door to many other domains of inquiry. In particular, it is used pervasively in philosophy, linguistics, mathematics, and computer science. Here are a few glimpses of some of the vistas that lie beyond, and some advice about where to look to find out more.

Philosophy

Philosophy and logic have always been closely knit together. The Greek philosopher Aristotle (384 - 322 BC) is usually said to be the inventor of logic. And to this day, philosophers are centrally involved in the development of the field, as philosophical concerns remain a driving force of innovation in logic. An up-to-date and accessible general survey of the main applications of logic in contemporary philosophy is:

> Ted Sider, *Logic for Philosophy*. Oxford University Press 2010.

The book contains easy-going introductions to free logic, modal logic and non-classical logic, and explains how they're applied in philosophical debates.

Paradoxes

Philosophers love paradoxes, and they pervade every area of philosophy. There are many ways to characterise what a paradox is. Here is one: a *paradox* is a seemingly sound argument for what appears to be a blatantly false conclusion. Many famous paradoxes originate with the ancient Greeks: a particularly rich source of them is the philosopher Eubulides of Miletus, who lived in the 4th century B.C. in present-day Turkey. He is said to be the inventor of the famous *Liar Paradox*. The liar paradox results from the consideration of the following sentence:

> (L): The sentence (L) is false.

One version of the paradoxical argument goes as follows:

If the sentence (L) is true, then (L) is true.	(tautology)
If (L) is true, then (L) is false.	(since (L) says of itself that it is false)
If (L) is false, then (L) is true	(using bivalence)
If (L) is false, then (L) is false.	(another tautology)
∴ (L) is both true and false.	(proof by cases)

The conclusion of this argument is a direct violation of the Law of Non-Contradiction.

Another famous paradox, which is also credited to Eubulides, is the Sorites paradox. 'Sorites' is the Greek word for heap, and it is so-called after the classical version of the paradox:

> A million grains of sand constitute a heap.
> When you remove single grain from a heap of sand, you are still left with a heap.
> ∴ One grain of sand constitutes a heap.

There are endlessly many variants of this argument. Predicates that can be used to form sorites arguments are called *vague* predicates — in this version the vague predicate in question is 'x is a heap'. Other typical examples of vague predicates are 'x is red', 'x is bald', 'x is rich', etc. Some philosophers believe all English predicates are vague.

A *resolution* of a paradox is an explanation for why the paradoxical argument is unsound in spite of seeming sound, or why the conclusion is true in spite ocf seeming false. Thus a crucial component in any discussion of a paradox is the question of the validity of the paradoxical argument, and in order to address this question rigorously we need to regiment the the argument by formalizing it into a formal language. If we formalize the liar and sorites arguments displayed above into $\mathscr{L}_0$ and $\mathscr{L}_1$ respectively, the result in both cases will be valid. Some logicians think that logic has to be revised in order avoid the conclusion that these arguments are actually valid.

The philosophical subject most closely associated with paradox is metaphysics. Much of metaphysics derives from the discussion of paradoxical puzzles. The interconnections between logical and metaphysical issues are so numerous that has recently become popular to think that doing logic and doing metaphysics are often just one and the same thing. (This view is particularly associated with the British philosopher Timothy Williamson).

But paradoxes are by no means unique to metaphysics; they occur in all areas of philosophy. Ethics has moral paradoxes, epistemology has paradoxes of rationality and the philosophy of science has paradoxes of confirmation. Many classic paradoxes concern the nature of space and time, and thus belong to the philosophy of physics. A short, entertaining and accessible introduction to the topic of paradoxes is the following:

Mark Sainsbury, *Paradoxes*. Third edition, Cambridge University Press 2009.

Non-classical logic

Many logicians believe that the quest to resolve paradoxes like the Liar and the Sorites motivates a revision of logic. For example, some philosophers hold that the liar argument I gave above is sound, and reject the Law of Non-Contradiction for that reason. Logics that do not assume Bivalence and/or Non-Contradiction are known as *non-classical logics*. In non-classical logic, some sentences have a status other than being simply true or false.

If a logic rejects non-contradiction, some of its statements are both true and false. In such logics, some formulae of the form $(\phi \wedge \neg\phi)$ are true. That means such logics need to invalidate *explosion* to avoid the conclusion that the moon is made of green cheese, say.

If a logic rejects bivalence, some of its sentences may have no truth value at all. The most influential logic of this kind is intuitionistic logic. Historically at least, intuitionism was motivated by the view that a statement is true just in case it can be verified, and false just in case it can be falsified. Take a statement like 'When he crossed the Rubicon, Julius Caesar had exactly 98,342 hairs on his head'; there is absolutely no way for us to verify or falsify what it says. According to many intuitionists, any such statement is neither true nor false.

There are ways of rejecting bivalence without holding that there are statements with no truth value. You might hold that some statements have truth values in between truth and falsehood, for example. The study of vagueness inspired the development of *fuzzy logics*, in which there are infinitely many truth values, modelled as real numbers between 0 and 1. Fuzzy logics have industrial applications, for example in household appliances. Rather suitably in view of their fuzzy contents, most modern washing machines run on fuzzy logic.

The standard textbook on the subject of non-classical logic is the comprehensive:

Graham Priest, *An Introduction to Non-Classical Logic*. Second edition, Cambridge University Press 2008.

The book is well-written, accessible and highly modular: you can use it to learn just the logic you're interested in, without reading all the preceding chapters. To elucidate all the various logics, it uses tree-style proof calculi, with which you are already familiar.

Modal logic

This course equips you to fully understand about 80% of the logic you are likely to meet in philosophy papers. Of the remaining 20%, the majority is probably *modal logic*, which is used in all core areas of philosophy. It differs from the logic we have done in this course in that it adds two new logical symbols to the language: '$\Box$' ('box') and '$\Diamond$' ('diamond'). The formation rules associated with these symbols are simple: you can stick them in front of any formula ϕ to obtain a new formula $\Box\phi$ or $\Diamond\phi$. If you add these rules to the syntax of $\mathscr{L}_0$, you get the language of standard modal logic; if you add them to the syntax of $\mathscr{L}_1$, you get quantified modal logic. The roots of modal logic go far back, but the formalism we presently use is pretty new. It was invented by the brilliant logician-philosopher Saul Kripke, who introduced it in a series of epoch-making papers, the first of which he wrote at the age of 17.

What do these new symbols mean? Well, the beauty of it is that they do not always have the same interpretation; they can mean various different things. The most common interpretation takes the '$\Box$' to stand for 'It is necessarily the case that' and '$\Diamond$' to stand for 'It is possible that'. For example, if P stands for 'It rains tomorrow', then '$(\Diamond P \land \Diamond\neg P)$' means 'It is possible that it will rain tomorrow and possible that it won't' or 'Whether or not it rains tomorrow is contingent'. '$\Box P$' means 'It is necessary that it will rain tomorrow', or 'No matter how things turn out, it will rain tomorrow'. '$\Box\neg P$' means 'It is impossible that it will rain tomorrow'. Using this interpretation of '$\Box$' and '$\Diamond$', we can use modal logic to speak about what could and could not happen, about what is, objectively speaking, possible and impossible. Since this topic is of great interest to metaphysics, it is called *metaphysical modality*, and on this interpretation '$\Box$' and '$\Diamond$' are said to stand for *metaphysical necessity* and *metaphysical possibility* respectively.

We can also apply modal logic to the investigation of *epistemic modality*. To do that, we take the '$\Box$' to stand for 'Certainly' or 'We know that' and the '$\Diamond$' to stand for 'Maybe' or 'For all we know, it may be the case that'. Now '$(\Diamond P \land \Diamond\neg P)$' means 'Maybe it will rain tomorrow, maybe it won't' or 'We don't know whether it will rain tomorrow' and '$\Box P$' means 'It will certainly rain tomorrow' or 'We know it will rain tomorrow'. This interpretation is importantly different from the previous one. No matter how things turned out, it would still have been true that $57 + 46 = 103$. So it is metaphysically necessary that this is so. But it needn't be epistemically necessary, since one can be uncertain what the sum of 57 and 46 is, or mistakenly believe it to be 105. Aside from epistemic modality, epistemologists also study *doxastic modality*, which is the belief-analogue of epistemic modality: '$\Box$' stands for 'We believe that' and '$\Diamond$' for 'it is compatible with what we believe that'.

Ethicists are interested in *deontic modality*. There, we take the '$\Box$' to mean 'ought to' or 'it is obligatory that' and '$\Diamond$' to mean 'may' or 'it is permitted that'. Let 'Q' stand for 'You will do your homework by tomorrow'. Then '$\Box Q$' means you *must* do your homework by tomorrow, '$\Diamond Q$' means that you are permitted to do it by tomorrow, and '$\Box\neg Q$' means you're not allowed to do it by tomorrow.

In the logic of *temporal modality*, '$\Box$' means 'always' and '$\Diamond$' means 'sometimes'. The list goes on — there are so many types of modality we can't mention them all here. Note that in all cases, '$\Diamond$' is equivalent to '$\neg\Box\neg$': a blizzard is possible iff it isn't necessary that there won't be one; smoking is permitted iff one is under no obligation not to smoke; it's sometimes dry iff it doesn't always rain, etc.

Although the Kripkean framework for doing modal logic has the resources to deal with all these different types of modality, we do not treat them all in the same way. For example, whether we want '$\Box P$' to entail 'P' depends on the type of modality we are talking about. 'It's always foggy' entails 'It is foggy', and 'Electrons necessarily repel one another' entails 'Electrons repel one another'. But the fact that you ought to do your homework does not entail that you will, and neither does the fact that you believe you will do your homework. So while we want '$\Box P$' to entail 'P' in the logics for temporal, epistemic and metaphysical modality, we do not want this entailment to hold in the logics for deontic and doxastic modality.

For this reason, there is a range of subtly different semantic treatments of modal logic available. The most common systems are called K, D, T, $S4$ and $S5$. Different modal logics are suited to different modalities, and sometimes it is a matter of controversy which one is best. For example, if you're obliged to do something, does it follow that you are also permitted to do it? That is, should the inference from '$\Box\phi$' to '$\Diamond\phi$' be valid in the logic of deontic modality? It's debatable (and hotly debated). In the modal logic D, which is usually associated with deontic modality, this inference is valid.

When we turn to quantified modal logic, further complexities arise. Here's an example. Barack Obama has no sons, only daughters, and things will likely stay that way. But plausibly, it was not impossible for him to have had a son: if a different sperm cell had won the race, he would have had one. If we let O stand for the property of being Obama's son, we can formalize this as $\Diamond\exists x Ox$: in some possible world, someone is Obama's son. Supposing this is true, does it follow that in the actual world, there is someone (or something) that would have been Obama's son, had things gone differently? That is, does it follow that $\exists x\Diamond Ox$? Intuitively, the answer is 'no', but many philosophers have argued it should be 'yes'.

The most prominent of these is the great logician Ruth Barcan Marcus. The principle that $\Diamond\exists x\Pi x$ entails $\exists x\Diamond\Pi x$ is known as the Barcan Formula.* This formula is the subject of a lively current debate in metaphysics. Those who reject it tend to argue that existence is contingent: the fact that you and I exist is partly a coincidence. If for any reason our parents hadn't met, we would never have come into existence. Supporters of the Barcan formula deny this. For example, Timothy Williamson holds that under such circumstances you would still have existed (although, as he puts it, you would never have become 'chunky').

Linguistics and the Philosophy of Language

In developing his logic, Frege was primarily driven by a desire to answer questions about the nature of mathematics. But he saw deep connections between mathematics and logic, and deep connections between logic and language. And it turned out that studying mathematics and language in conjunction greatly benefits our understanding of both. As a result, Frege's contributions to the understanding of language are as lasting and influential as his contributions to the foundations of mathematics.

One of Frege's most important contributions is the invention of the basic trichotomy between expressions, intensions and extensions that we have used throughout this course. This

*The Barcan formula is the statement: $(\forall x\Box\Pi x \supset \Box\forall x\Pi x)$. Its contrapositive is $(\Diamond\exists x\Pi x \supset \exists x\Diamond\Pi x)$.

trichotomy still survives (be it in somewhat altered form) in the current practice of linguistics. Frege's word for extension was *Bedeutung* (translated 'reference'), and his word for intension was 'Sinn' (translated 'sense'). His original presentation of the distinction, along with many other deep insights, can be found in his seminal paper *On Sense and Reference*, which is essential reading for any intro to philosophy of language or intro to semantics course:

> Gottlob Frege, "On *Sinn* and *Bedeutung*". In Michael Beaney (ed.), *The Frege Reader*, Blackwell 1997.

The relationship between Natural and Formal languages

Throughout the course, we have continually emphasised that the formalization of an English sentence into our formal languages often achieves no more than an approximation of the meaning and the truth conditions of the original. Many deep and interesting issues in philosophical semantics arise from asking the question what differences remain. Thus there is the question whether the truth-functional connectives '$\neg$', '$\wedge$', '$\vee$' and '$\supset$' capture the full meaning of their English analogues 'not', 'or', 'and' and 'if . . . then'.

In order to address these questions properly, it is crucial to distinguish the actual meanings of expressions from the implicatures and what Frege called the *colour* of words. The words 'house' and 'habitation' differ in colour, but that doesn't mean they're not synonymous. In today's terminology, we must distinguish *semantic* questions from *pragmatic* ones.

Let me give an example to illustrate this. Johnny is a quiet and kindly man in his early 50s who loves mother very much (she is still alive). He has some trouble with stomach acid, and he takes tablets every day to help with that. Now ask yourself if it would be appropriate to describe that situation using the following sentence:

Either Johnny takes his pills every day, or he will kill his mother with a chainsaw. (1)

Obviously, that would not be appropriate. (1) erroneously suggests that Johnny is a psychopathic chainsaw murderer, only just about held in check by his medication. One could use (1) to justify extreme measures to ensure Johnny gets his pills every day, but in fact no such measures are warranted. For all these reasons, (1) is a very misleading characterisation situation described, and for that reason one might be inclined to think is false.

However, if we suppose the English 'or' is a truth-functor, expressing the same truth-function as '$\vee$', then (1) is perfectly *true*, since the first disjunct is true: John does take his pills every day. If, in spite of this, we stick with the judgment that (1) is false, we must reject the truth-functional analysis of the English 'or'. (Perhaps English disjunctions require not only the truth of one of their disjuncts to be true, but also some connection between them.) For a long time, a lot of philosophers and logicians were content to accept counterexamples a lot like this one. But today almost any linguist or philosopher of language will tell you that the judgment that (1) is misleading offers no substantial evidence against the simple truth-functional analysis of 'or'.

Why not? Well, not every misleading statement is false. There is a Monty Python sketch which illustrates this well. In the sketch, two bored pilots in a perfectly functional passenger airplane decide (for no reason) to tell the passengers 'There is absolutely no reason to panic.' Encouraged by the consternation this remark immediately causes, they go on to announce 'The wings are still securely attached to the aircraft and the engines are not about to catch fire.' Of course, everything these pilots say is strictly speaking *true*: the wings *are* securely attached to the aircraft and there *is* no reason to panic. However, simply by mentioning this, the pilots misleadingly implicate that there is some kind of serious problem.

Returning to (1), it is easy to explain *why* the sentence is misleading, *even on the assumption that it is true*. Aside from expressing a proposition that may be true or false, when I assert (1), I raise the possibility of Johnny mowing down his mother with a chainsaw. The very fact that I take the trouble to raise this possibility suggests that this is somehow a live concern, even if that isn't part of the literal content of what I said. So there may be a perfectly satisfactory explanation for why sentences like (1) are misleading that does not depend on their being false. There is no need, then, to abandon the attractively simple truth-functional analysis of 'or' in favour of a more complex semantic theory.

In the case of other connectives, we can say similar things. Consider this sentence:

If Johnny doesn't take his pills every day, he will kill his mother with a chainsaw. (2)

Most people have the intuition that when we formalize this sentence using the material conditional, we are leaving out something important: namely the connection between the antecedent and the consequent. We just saw there is reason to think that the connection between the disjuncts of (1) is merely implicated, and not part of what (1) literally says. Similarly, many have argued that the connection between antecedent and consequent is not part of what (2) literally says.

When assessing the truth conditions of English sentences, then, it is very important to distinguish a situation in which the sentence is merely misleading from a situation in which the sentence is false. In particular, many purported counterexamples to the truth-conditional analyses of English connectives can be seen to fail when we pay careful attention to this distinction. It is often far from easy to know what is literally said and what is merely implicated. The philosopher of language Paul Grice developed the modern framework for thinking about these questions systematically. The most important of these is the following paper, in which Grice addresses the question of the adequacy of the kinds of formalizations used in this course:

> Paul Grice, "Logic and Conversation". In Grice, *Studies in the Ways of Words*, Harvard University Press 1991.

Conditionals

In the last section, we briefly touched on the analysis of English conditionals. The search for a correct analysis of natural language conditionals plays a major role in both philosophy and linguistics, and conditional claims play a the crucial and apparently ineliminable role in all areas of inquiry.

Natural language conditionals are usually divided up into *indicative* and *counterfactual* conditionals:

1. If Oswald didn't shoot Kennedy, then someone else did. (indicative)
2. If Oswald hadn't shot Kennedy, then someone else would have. (counterfactual)

Almost everyone agrees that (1) and (2) have radically different meanings: while (2) suggests that there was someone other than Oswald plotting Kennedy's demise, (1) does no such thing. Roughly speaking, counterfactual conditionals tend to have antecedents that are presumed to be false, while indicative conditionals have antecedents whose truth is presumed to be merely unknown or in question.

The material conditional, i.e. the truth-function associated with '$\supset$', is not usually considered a serious candidate for the analysis of counterfactual conditionals. Since these conditionals usually have false antecedents, the material conditional would render almost any of them true. But the material conditional does have something going for it as an analysis of indicative conditionals, and some very strong arguments have been proposed in its favour, including some purported proofs that the material and indicative conditionals must be logically equivalent.

If you're curious about the relationship between natural language conditionals and the material conditional, have a look at the first three sections of the following paper by Angelika Kratzer, who is the world's leading authority on conditionals. They contain a very readable overview of the arguments in favour and against the material conditional analysis.

> Angelika Kratzer, "Conditionals". In Kratzer, *Modals and Conditionals*, Oxford University Press 2012.

(The other sections of the paper are a bit more technical, defending her own analysis of the way in which the conditional analysis fails, and the thesis that 'the history of conditionals is the story of a syntactic mistake'.)

Much of the philosophical discussion of *counterfactual* conditionals has focussed around the influential possible worlds analysis of counterfactuals, proposed independently (and in slightly different forms) by David Lewis and Robert Stalnaker. The classic presentation is Lewis's highly readable and short monograph:

> David Lewis, *Counterfactuals*, Blackwell 1973.

Definite descriptions

Another much-debated question concerns the adequacy of the Russellian analysis of definite descriptions. Russell's analysis of definite descriptions has had tremendous influence in many areas of philosophy. It has been thought to provide the key to solving a host of metaphysical puzzles surrounding non-existence. In addition, many believe that it illuminates foundational issues concerning our ability to refer to objects in the external world.

For much of the 20th century, it was thought that Russell's treatment of definite descriptions could be extended to cover proper names like 'John' and 'Aristotle' as well. Roughly speaking, the name 'Aristotle' was thought to be synonymous with a complex definite description involving properties commonly attributed to Aristotle. However, most philosophers now take the view that Saul Kripke decisively refuted that theory of proper names in his seminal work *Naming and Necessity*.

The analysis was originally proposed in Russell's classic 1905 paper *On Denoting*. More recently, the adequacy of the analysis has been the subject of an influential debate between Keith Donnellan and Saul Kripke, which has done much to clarify the merits of Russell's analysis. All three papers are contained in this anthology:

> Gary Ostertag (ed.), *Definite Descriptions: A Reader*. MIT Press 1991.

Dynamic Predicate Logic

When formalizing English sentences into predicate logic, perhaps you sometimes felt that $\mathscr{L}_1$ was somewhat maladapted to the purpose, because it just doesn't work the way English does.

Let me illustrate this with an example. When formalizing a sentence like

$$\text{There's a witch and she lives in the forest} \tag{3}$$

one is naturally inclined to write down

$$(\exists x Wx \land Fx) \tag{4}$$

This is incorrect, because the existential quantifier in (4) does not bind the variable in the second conjunct. Thus it is an open formula without a truth value. The correct way to formalize (3) into $\mathscr{L}_1$ is as follows:

$$\exists x(Wx \land Fx) \tag{5}$$

This way, both x's are inside the scope of the quantifier, and now we have a closed formula that mimics the truth conditions of (3).

But the choice for (4) is a very understandable mistake, resulting from well-founded intuitions about the syntax of (3), the English statement we began with. For while (3) and (5) have the same truth conditions, they do not share the same syntactic structure. (5) is an existential statement. It *contains* a conjunction, but that conjunction occurs entirely within the scope of the quantifier, and so the statement as a whole is no conjunction. By contrast, (3) has 'and' as its main connective, and so it *is* a conjunction consisting of two separate clauses, only the first of which contains an existential quantification. In short, the syntactic structure of (3) is precisely analogous to that of (4).

Similarly, the natural way to formalize the sentence (6) would be to use the formula (7). But again, this isn't right because (7) contains a free variable. Instead we have to formalize (6) as (8).

$$\text{If a witch has a broomstick, she also has a cat} \tag{6}$$

$$(\exists x(Wx \land Bx) \supset Cx) \tag{7}$$

$$\forall x((Wx \land Bx) \supset Cx) \tag{8}$$

Both in the case of (3) and in the case of (6), the issue seems to be that while the English quantifiers can bind variables outside of their own scope, the quantifiers of $\mathscr{L}_1$ cannot do this. This raises the question whether it is possible to re-engineer the semantics of predicate logic so that it becomes like English in this respect.

A few decades ago, two Amsterdam logician-philosophers asked themselves precisely this question, and they found out that the answer is 'yes'. The logic they developed is called *Dynamic Predicate Logic*. The language of dynamic predicate logic is syntactically identical to $\mathscr{L}_1$ (that is, it has all and only the same formulae). But the semantics is different. In particular, (4) and (7) are sentences in this language, with truth values. And they intuitively have the right truth conditions: in this system they are logically equivalent to (5) and (8) respectively.

You can read all about this system in their paper:

> Jeroen Groenendijk and Martin Stokhof, "Dynamic Predicate Logic". In *Linguistics and Philosophy* 14.1, 1991.

By providing a formal language whose quantifiers mimic the binding behaviour of English quantifiers, this paper brings us much closer to understanding how binding works in English and other natural languages. Incidentally, this work also bears on another topic we just talked about: viz., the analysis of conditionals. Groenendijk and Stokhof use a conditional that is not truth-functional. The fact that they need this alternative conditional in order to describe the binding behaviour of quantifiers embedded in conditionals is good evidence that natural language conditionals are not in fact truth-functional.

Mathematics and Computer Science

Frege tried to use his logical system to show that a great number of mathematical facts are in fact logical truths. In particular, he wanted to show that number theory could be entirely subsumed under logic. Today, this position is known as *logicism*. Logicism stands opposed to the notion that arithmetical reasoning depends on a special kind of human intuition, as Kant thought, and also to John Stuart Mill's contention that mathematical truths are empirical and contingent, in the same way that the laws of nature are. In Set VII, we have already seen a few of the key techniques involved in the effort to characterise number talk in logical terms.

In 1884, Frege published *The Foundations of Arithmetic*. This is a highly readable and accessible little book. (It doesn't even require the background of this course, although that background will give you a better appreciation of what's going on). The book opens with a strident and witty critique of the views of Kant and Mill, and then proceeds to outline and defend Frege's own views on the nature of number. Short as it is, the *Foundations* abounds with crisp and compelling arguments and highly original ideas, all presented with remarkable clarity and rigour. Whether or not they agree with Frege's views, many analytic philosophers still regard this remarkable work as the ideal example of what a philosophical text should strive to be.

Gottlob Frege, *The Foundations of Arithmetic*. Translated by J.L. Austin, Northwestern University Press 1980.

Higher-Order Logic

George Boolos, *Logic, Logic and Logic*. Harvard University Press 1999.

Axiomatic Set Theory

Herbert B. Enderton, *Elements of Set Theory*. Academic Press 1977.

Gödel's Theorems

Peter Smith, *An Introduction to Gödel's Theorems*. Second Edition, Cambridge University Press 2013.

Computability

George Boolos and Richard Jeffrey, *Computability and Logic*. Cambridge University Press, 1989.