Mathematical Fictionalism

DHOEK@PRINCETON.EDU, WWW.DANIELHOEK.COM, 20 NOVEMBER 2019

> "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."
>
> — Eugene Wigner, *The Unreasonable Effectiveness of Mathematics*

> "By focussing on the question of the application of mathematics to the physical world, I was led to a surprising result: that to explain even very complex applications of mathematics to the physical world (for instance, the use of differential equations in the axiomatization of physics) it is not necessary to assume that the mathematics that is applied is true."
>
> — Hartry Field, *Science Without Numbers*

The Quinean Challenge and the Way of the Weasel

Quine self-described as a nominalist *manqué*, and "a reluctant platonist only in honest recognition of what have seemed to be the demands of science."[1] In spite of his famous preference for desert landscapes, he found himself forced to inflate his otherwise sparse ontology by admitting the existence of an infinity of mathematical objects like numbers and sets. After all, our best scientific theories quantify over such entities, and "to be is to be the value of a bound variable." In his paper "Whither Physical Objects," Quine considers the possibility of eliminating out the physical objects and keep *only* the mathematical ones, but he did not see any way to eliminate the mathematical objects.

You can think of Quine's argument as a challenge to the nominalist: "You say it is possible to explain everything the scientists explain, but without making any reference to mathematics. Well then *do* it!" Quine was a platonist because he thought that this challenge could not be met. In his *Science Without Numbers*, Hartry Field started a program of actually meeting this challenge, formulating Newton's theory of gravity in a way that made reference to , but not to numbers, functions or derivatives (see also Arntzenius and Dorr 2011, Chen 2018, as well as others listed below).

But contemporary nominalists on the whole prefer to reject the Quinean challenge altogether, seeking instead to walk what Joseph Melia called *the way of the weasel*, also known as *cheap* fictionalism (Melia 2000, Yablo 1998, 2005, and others listed below). Rather than doing the hard work of writing down a

[1] Letter to Hartry Field, reproduced in the new edition of *Science Without Numbers*.

nominalistically physical theory that makes no reference to numbers, weasely nominalists propose a lazy short-cut. Simply take your favourite physical theory T, formulated the normal way, with reference to mathematics, and replace it with $\circlearrowleft T$:

$$\circlearrowleft T \;=\; T\text{, except that mathematical objects might not exist.}$$

More ambitious weasels like Yablo and y.t. are not only concerned with the wholesale nominalisation of entire theories, but also individual propositions involving mathematical objects:

$$\circlearrowleft \phi \;=\; \phi\text{, except that mathematical objects don't exist.}$$

In some cases at least, $\circlearrowleft \phi$ can be (approximately) reformulated without appeal to subtraction:

	p	$\circlearrowleft p$
(1)	p_1: *The number of my socks is three.*	r_1: *I have a sock and another and another and that's it.*
(2)	p_2: *No kiwi has an even number of seeds.*	r_2: *One can never pair up all the seeds of any kiwi: you will always be left with a single unpaired seed.*

To use Yablo's image, the nominalist subtraction operator '$\circlearrowleft$' unwraps the concrete *nugget* from the mathematical swaddling in which the has hidden it.

If successful, this more fine-grained approach to weaseling will have the advantage of helping us to understand how mathematical *language* works, and the role that mathematics plays in communication. Why is it that we make reference to mathematical objects not only when we are actually interested in mathematical questions, but also when we are describing the concrete world around us? As Yablo emphasises, this is a question that Field's program does *not* answer:

> "Field lays great stress on the notion that mathematical theories are *conservative* over the nominalistic ones, i.e., any nominalistic conclusions that can be proved with mathematics can also be proven (albeit often much less easily) without it. The utility of mathematics lies in the *no-risk deductive assistance that it provides to the beleaguered theorist*.
>
> This is on the right track, I think. But there is something strangely half-way about it. I do not doubt that Field has shown us a way in which mathematics can be useful without being true. It can be used to facilitate deduction in nominalistically reformulated theories of his own device … This leaves more or less untouched, however, the problem of how mathematics does manage to be useful without being true. It is not as though it benefits only practitioners of Field's qualitative science (it does not benefit Field-style scientists at all; there aren't any). The people whose activities we are trying to understand are practicing regular old platonic science.
>
> How, without being true, does mathematics manage to be of so much help to *them*?" (Yablo *The Myth of the Seven*, p. 91)

Mathematical Exculpature

The core objection to the way of the weasel is that $\circlearrowleft T$ is either contradictory or unintelligible:

> **Intelligibility Objection**: "if we cannot say what we want any other way except by weaseling, it is just not clear what we are saying. […] We can change the story we are narrating by adding or subtracting minor details, but we can hardly be thought to be telling a consistent story (or in some cases, any story at all) if we take back too much. In short, there are limits to how much weaseling can be tolerated. J. R. R. Tolkien could not, for example, late in the Lord of the Rings trilogy, take back all mention of hobbits." (Colyvan, p. 295)

We have seen this kind of skepticism about subtraction before. I hope that by now, most of us are a little bit more optimistic that logical subtraction can extend our expressive resources, and draw new lines in logical space. But even so, Colyvan's worry seems justified: on most theories of subtraction, $P - Q$ is not well-defined in all cases. How can we be sure $\circlearrowleft T$ is a subtraction of the right kind? Colyvan's analogy certainly seems to be apt: if anything references to mathematics are even more prevalent in the formulation of physical theories than references to hobbits in the *Lord of the Rings*.

To respond to this objection, we are going to need to explain in greater detail what the mapping $\circlearrowleft$ is. Last week we saw how the exculpature mechanism maps the proposition *Ellen was wearing the same sort of hat Sherlock Holmes* to the proposition *Ellen was wearing a deerstalker*, which makes no reference to Sherlock Holmes at all. Can we do the same with numbers?

Proposal. The mapping '$\circlearrowleft$' between intensional propositions is $p \mapsto C(p \upharpoonright m)$, where the subject matter C is *the concrete world* and m is the relevant "mathematical myth". If, as in our examples above, p only makes reference to natural numbers, m can be specified as follows:

> Beyond the outer reaches of our physical universe, there is the Platonic Realm of Mathematics. Amongst the denizens of this land are the unchanging Natural Numbers, arranged on the Natural Number Line. All the way on the left sits the number Zero. Immediately to Zero's right sits One. To the right of One sits Two, and so on. To the immediate right of every natural number sits another natural number. Every natural number numbers the class of natural numbers seated to its left and all and only classes equinumerous to that class. The End.[2]

---

[2] This myth is formulated in a second-order language, and it makes reference to classes. For our purposes here, we only need the following very simple theory of classes: 1. There is a class containing nothing; 2. For any object x, there is a class containing x and nothing else; 3. For any two classes A and B there is a class containing all and only the objects contained in A and B. (We don't even need extensionality.) Depending on whether you are a nominalist about classes, you can write these classes into the platonic myth as well, or assume they are already part of the nominalistic universe. A class A is *finite* iff some injective function from A to A is not surjective. Two classes A and B are *equinumerous* iff there is a bijection between the objects in A and the objects in B.

Let's try and see how this fares with (1) above. To confirm that $\circlearrowleft p_1 = r_1$, we need to check three things:

- *Aboutness*: r_1 is about the target subject matter C.
  Clearly this condition holds, since r_1 turns only on the socks I possess, and its truth is completely independent of what goes on in the mathematical realm.
- *Equivalence*: p_1 is conditionally equivalent to r_1, assuming m (that is, $p_1 \restriction m = r_1 \restriction m$).
  Left-to-Right: if p_1 and m are both true, then there must be some bijection that maps the numerals to the left of Three (Zero, One and Two) onto all the socks I own. It follows that I have a sock, and another, and another, and that's it.
  Right-to-Left: suppose r_1 and m are both true. Then label my first sock S_{Zero}, the other one S_{One} and the final one S_{Two}. Then the map $S_n \mapsto n$ is a bijection between all my socks and the numbers to the left of Three. Hence by n it follows that the number of my socks is Three.
- *Independence*: m has no bearing on C.
  Since the mathematical realm of the numbers is outside the concrete world, every state of the concrete world (every cell of C) is compatible with both m and $\neg m$.

What about example (2)? Here we get *Aboutness* and *Independence* in the same way. That leaves only:

- *Equivalence*: p_2 must be conditionally equivalent to r_2, assuming m.
  We will prove the contrapositive: $\neg p_2 \restriction m = \neg r_2 \restriction m$. Suppose m is true. Now suppose $\neg p_2$, so that there is a kiwi with an even number of seeds — say the number is $2N$. Then there is a bijection $n \mapsto K_n$ from numbers to the left of $2N$ to kiwi sees, whence $\neg r_2$ since all the kiwi seeds can be paired up thus: $\{K_0, K_1\}$, $\{K_2, K_3\}$… $\{K_{2N\text{-}2}, K_{2N\text{-}1}\}$. Conversely if $\neg r_2$, and there is some kiwi whose seeds can all be paired up, then we can associate pairs of seeds with pairs of numbers $\{0, 1\}$, $\{2, 3\}$, … $\{2N\text{-}2, 2N\text{-}1\}$, so that $\neg p_2$ because there is a kiwi with $2N$ seeds.

Sidenote. In example (2), $\circlearrowleft$ maps a proposition that is *true* by the nominalist's lights (because there are no numbers) to a proposition that is *false* by anyone's lights (the seeds of about half of all kiwis can be paired up). So if we want to think of $\circlearrowleft p$ as $p - m$, then we will need a theory of subtraction that allows $x - y$ to be stronger than x in cases where y is not entailed by x (as the theory of exculpature does; see also Yablo 2014, ch. 11). If we want to maintain the view that the remainder and the subtracted proposition are entailed by the whole, we can do that by instead analysing $\circlearrowleft p$ in the following way: $(m \wedge p) - m$. But then we have to set aside the idea of subtraction as the inverse of conjunction, since we cannot allow it to be the case that $\circlearrowleft p = (m \wedge p) - m = p$.

Explanation and Subtraction

The second objection is that, even if the nominalised claim $\circlearrowleft\phi$ were to make sense, it would still not have the same explanatory purchase as ϕ does:

> **Explanatory Objection**: "let us grant that metaphorical language (and figurative language generally) can be used for purposes of true description, as Walton and Yablo argue. The important question for our purposes is whether figurative language can be explanatory" (Colyvan p. 299)

Colyvan goes on to concede that metaphors can be explanatory as long as there is a suitable, non-metaphorical substitute available. Nonetheless he thinks there are special problems for the mathematical case, because mathematical explanations make an *essential* appeal to mathematical entities and mathematical theory — they cannot be "swapped out" for concrete substitutes. He give the example of "Kirkwood gaps" in the asteroid belt to make this point:

> The Kirkwood gaps are localized regions in the main asteroid belt between Mars and Jupiter where there are relatively few asteroids. The explanation for the existence and location of these gaps is mathematical and involves the eigenvalues of the local region of the solar system (including Jupiter). The basic idea is that the system has certain resonances and as a consequence some orbits are unstable. Any object initially heading into such an orbit, as a result of regular close encounters with other bodies (most notably Jupiter), will be dragged off to an orbit on either side of its initial orbit. An eigenanalysis delivers a mathematical explanation of both the existence and location of these unstable orbits … The explanation of this important astronomical fact is provided by the mathematics of eigenvalues (that is, basic functional analysis)." (Colyvan, p. 295)

Colyvan's point here is not that the mathematical theory of resonances etc. is by itself a *complete* explanation of the position of these Kirkwood gaps. For example, if Jupiter had a different mass, then the gaps would be in different places. So to work out the particular locations of the gaps, we also need concrete assumptions about this particular system. Rather, Colyvan's point here is that the mathematics plays an ineliminable role in the explanation.

Here is a different way to frame Colyvan's worry. To derive empirical predictions from their theories, physicists have to do a lot of elaborate reasoning and calculation. That reasoning makes essential appeal to mathematics, in that the arguments involved would be invalid if the mathematical premises were omitted. Consequently, the truth of those empirical predictions is apparently not guaranteed by our physical theory unless we assume that the mathematical assumptions to which we appealed are also true. And insofar as those predictions are borne out by observation, the theory does not (fully) explain those observations except with the help of those mathematical assumptions.

To drive the Colyvan point home, consider for example the following prediction/explanation:

(3) This glass contains a solution of phenolphthalein in water

(4) I will add sodium carbonate to the glass and stir.

(5) [Some chemistry describing the interaction between these two chemicals]

∴ (6) The water will turn pink.

Now suppose I subtract the phenolphthalein from (3), to get (3*):

(3*) This glass contains water [without any phenolphthalein]

What we are left with after the subtraction is no longer a full explanation for (6). And the premises (3*), (4) and (5) do not support the prediction that the water will turn pink.

Colyvan is right to worry that subtracting the mathematics will lead to such losses of explanatory power: as we'll discuss next week, there are nominalisation strategies that do not guarantee the preservation of validity. But as it turns out, the present theory of exculpature does guarantee this:

Preservation of Validity For any propositions $p_0, p_1 \ldots$ and c such that $\circlearrowleft p_i$ and $\circlearrowleft c$ are well-defined, the following holds: if $p_0, p_1 \ldots \vDash c$ then $\circlearrowleft p_0, \circlearrowleft p_1 \ldots \vDash \circlearrowleft c$.

Corrollary. If $\circlearrowleft p_0, \circlearrowleft p_1 \ldots \vDash c$ and the conclusion c is wholly about C (that is, if c is wholly about concrete, non-mathematical matters), then $\circlearrowleft p_0, \circlearrowleft p_1 \ldots \vDash c$.

Proof outline. To see intuitively why these results hold, first note that if $p_0, p_1 \ldots \vDash c$, then the intersection of the premises, thought of as an area of logical space, is included in the conclusion. So this inclusion must also hold as restricted to the platonist m-worlds: the area where $p_0{\upharpoonright}m$, $p_1{\upharpoonright}m$ and so on are true is included in the area where $c{\upharpoonright}m$ is true. The completion by C essentially inflates each partial proposition to cover all of logical space, while retaining their relative logical 'shapes'. Thus it still preserves the inclusion, and the region where the loose readings of he premises $C(p_0{\upharpoonright}m)$, $C(p_1{\upharpoonright}m)$ and so on are all true is included in the $C(c{\upharpoonright}m)$-region. So $C(p_0{\upharpoonright}m), C(p_1{\upharpoonright}m) \ldots \therefore C(c{\upharpoonright}m)$ is a valid entailment. In the particular case where c is concrete, and thus *already* wholly about C, we have $\circlearrowleft c = c$ whence $\circlearrowleft p_0, \circlearrowleft p_1 \ldots \vDash c$ is valid too.

Consequently, any concrete predictions delivered by a partly mathematical theory in physics or any other science also follows from the nominalised reading of the premises. And checking the literal validity of a partly mathematical argument may just be the most efficient way we have to verify the validity of the nominalised analogue of that derivation.

A few stray observations on this:

- This formal result is closely linked to Hartry Field's observation that mathematical language makes for a *conservative* extension over any nominalist theory, which is to say that adding the mathematical theory onto the nominalist theory will not allow us to draw any new conclusions.
- This same formal result also explains why we find metaphorical arguments like the following unobjectionable:

p_A : Lazio is in the knee of the Italian boot, and Calabria is in the toe.

p_B : The knee of the Italian boot is north of its toe.

∴ c : Lazio is north of Calabria.

- In the chemistry example above, we saw that "subtracting" the phenolphthalein did not preserve the validity of the argument (3-6). That means that this subtraction can't have been the result of uniform exculpature. It also means that there is no "easy road" to anti-realism about phenolphthalein through exculpature. Next class, we will use such considerations to think about which types of subtraction-based, easy anti-realism are and are not tenable.

Mathematical Necessity

The account of mathematical exculpature makes reference both to possible worlds in which numbers do not exist, and to worlds in which they don't. The trouble is that both platonists and nominalists have traditionally been in agreement that such purely mathematical propositions are either necessarily true or necessarily false. We might call that doctrine *mathematical fatalism*.

Contingentist Strategy: Reject mathematical fatalism. Positive arguments for contingentism (delivered to us from the *Q* via Gideon Rosen's distant cousin):

- The Conceivability argument
- The Humean/Lewisian argument
- The Strong Consistency argument

Note in addition that exculpature provides the contingentist with a neat error theory: for any purely mathematical proposition p entailed by m, $\circlearrowleft p = \top$. And for any purely mathematical proposition p incompatible with m, $\circlearrowleft p = \bot$. Thus exculpature explains why fatalism seems so plausible: the loose (nominalised) readings of purely mathematical statements *do* have their truth-value essentially, even if the propositions they literally express do not.

Instrumentalist Strategy: Allow our semantic/pragmatic theory to make reference to scenarios that are not metaphysically possible. "Granted, these scenarios can't both be metaphysically possible. But to refuse help from them *for that reason* in our semantic theorizing seems like an excessive show of Kripkean piety."

Squumber Substitution Strategy: Sure, the myth m is metaphysically impossible. But now replace the numbers in the story with *squirrels*, arranged on an infinite *squirrel line* (or, if you will, a squumber line). This gives us a scenario m^* that's only mildly more ridiculous than the original, which is blatantly contingent, and which can do all the work m can do.

Bibliography

Honest Toil

Arntzenius, Frank, and Cian Dorr, 2011, "Calculus as geometry." Chapter 8 of Frank Arntzenius, *Space, Time, and Stuff*. New York: Oxford University Press.

Burgess, John, 1984, "Synthetic Mechanics." *Journal of Philosophical Logic* 13: 379-95.

Chen, Eddy, 2018, "An Intrinsic Theory of Quantum Mechanics: Progress in Field's Nominalistic Program." Manuscript, https://arxiv.org/abs/1810.06551.

Colyvan, Mark, 2010, "There is No Easy Road to Nominalism." *Mind* 119(474): 285-306.

Field, Hartry, 2016, *Science Without Numbers: A Defense of Nominalism*. Second Edition. New York: OUP.

Meyer, Glen, 2009, "Extending Hartry Field's Instrumental Account of Applied Mathematics to Statistical Mechanics." *Philosophia Mathematica* 17 (3):273-312

Mundy, Brent, 1987, "The Metaphysics of Quantity." *Philosophical Studies* 51: 29-54.

Theft

Azzouni, Jody 1997: "Applied Mathematics, Existential Commitment and the Quine-Putnam Indispensability Thesis." *Philosophia Mathematica* 5: 193–209.

Burgess, John and Gideon Rosen, 1997, *A Subject with No Object*. Oxford: Clarendon Press.

Chihara, Charles, 1990, *Constructibility and Mathematical Existence*. New York: Oxford University Press.

Dorr, Cian, 2010, "Of Numbers and Electrons." *Proc. of the Aristotelian Society* 110(2.2): 133-181.

Hellman, Geoffrey, 1989, *Mathematics Without Numbers*. Oxford: Oxford University Press.

Melia, Joseph, 2000, "Weaseling Away the Indispensability Argument." *Mind* 109: 466-71.

Yablo, Stephen, 1998, "Does Ontology Rest on a Mistake?" *Proc. of the Aristotelian Society* 72 (1):229-262.

2005, "The Myth of the Seven." In *Fictionalist Approaches to Metaphysics*, ed. Mark Eli Kalderon, 88-115. New York: Oxford University Press.